Chapter 3

Fourier Series

In this chapter we are concerned with the formation of series based on orthogonal sets of functions, primarily orthogonal sets of trigonometric functions. This series is named after the French mathematical physicist Joseph Fourier (1768 – 1830).

3.1 Piecewise Continuous Functions (PWC)

Definition 3.1.1: A function $f$ is piecewise continuous in $[a,b]$ if it is continuous at all points on the interval except at most at a finite number of points where finite discontinuity may exist. We assume that $f$ is real valued function of a single variable $x$.

Notation:

✓ The left hand limit of $f$ at $x_0$ is

$$ f(x_0^-) = \lim_{x \to x_0^-} f(x) = \lim_{h \to 0} f(x_0 - h) $$

✓ The right hand limit of $f$ at $x_0$ is

$$ f(x_0^+) = \lim_{x \to x_0^+} f(x) = \lim_{h \to 0} f(x_0 + h) $$

Definition 3.1.2

1) $f$ is continuous at $x_0$ if $f(x_0^+) = f(x_0^-) = f(x_0)$.

2) If $f(x_0^+), f(x_0^-)$ are unequal but both exist, we say that there is a jump discontinuity at $x_0$. The jump is defined as $f(x_0^+) - f(x_0^-)$.

3) If $f$ is PWC on $[a,b]$ then $f$ is bounded and integrable on $[a,b]$, that is $\exists M$ such that $|f(x)| \leq M \quad \forall x \in [a,b]$ and $\int_a^b f(x) \, dx$ exist.

4) If $f$ is PWC and $\{a, x_1, x_2, \ldots, x_{n-1}, b\}$ are points where $f$ is discontinuous then:

$$ \int_a^b f(x) \, dx = \int_a^{x_1} f(x) \, dx + \int_{x_1}^{x_2} f(x) \, dx + \ldots + \int_{x_{n-1}}^b f(x) \, dx. $$

5) If $f_1$ and $f_2$ are PWC functions on $[a,b]$ then $f_1f_2, c_1f_1 + c_2f_2, f_1^2$ are all PWC. Therefore the integral of these combinations over the interval must exist.

Example: Graph the function and determine whether it is PWC, continuous, and find its jump.

$$ f(x) = \begin{cases} 
  x + 1, & \text{if } 0 \leq x < 1; \\
  4, & \text{if } x = 1; \\
  x^2 - 1, & \text{if } 1 \leq x < 3; 
\end{cases} $$
Definition 3.1.3

1) Left hand derivative of a function \( f \) at \( x_0 \) is
\[
f'_-(x_0) = \lim_{h \to 0} \frac{f(x_0) - f(x_0 - h)}{h}.
\]

2) Right hand derivative of \( f \) at \( x + 0 \) is
\[
f'_+(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 +)}{h}.
\]

3) If \( f'_-(x_0) = f'_+(x_0) \) and \( f \) is continuous at \( x_0 \) then \( f \) is differentiable at \( x_0 \).

4) If \( f \) is PWC function with jump discontinuity at \( x_0 \) then the derivative fails to exist at that point.

5) \( f \) is smooth on \([a,b]\) if it has continuous derivative on \([a,b]\).

6) \( f \) is piecewise smooth (PWS) on \([a,b]\) if it is PWC and has a PWC derivative on \([a,b]\).

Examples:

Q1) Let \( f(x) = \begin{cases} x^2 - 1, & \text{if } x < 0; \\ x^2 + 1, & \text{if } x > 0. \end{cases} \)

Discuss continuity and smoothness of \( f \), compute its jump.

Q2) Let \( f(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases} \)

i) Find right and left hand derivatives for \( f \) at \( x = 0 \).

ii) Determine \( f'(0) \) if it exist.

Q3) Let \( f(x) = \begin{cases} e^{-x}, & \text{if } x \leq 0; \\ e^x, & \text{if } x \geq 0. \end{cases} \)

Find \( f'_+(0), \quad f'_-(0), \quad f'(0) \) if exist.
3.2 A Basic Fourier Series

From example 2.6 page 39 the eigenfunction of \( y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L) \) are 1 for \( \lambda = 0 \) and \( \sin \frac{n\pi x}{L}, \cos \frac{n\pi x}{L} \) for \( \lambda = \frac{n^2\pi^2}{L^2} \).

The series for a function \( f(x) \) is
\[
f(x) = \sum_{n=1}^{\infty} C_n g_n(x).
\]

\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx,
\]
\[
a_n = \frac{1}{\| \cos(n\pi x/L) \|^2} \int_{-L}^{L} f(x) \cos(n\pi x/L) \, dx
\]
\[
\| \cos(n\pi x/L) \|^2 = \int_{-L}^{L} \cos^2(n\pi x/L) \, dx = L.
\]
\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(n\pi x/L) \, dx, \quad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}.
\]

Similarly \( b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(n\pi x/L) \, dx, \quad n \in \mathbb{N} \).

The series is called a basic Fourier series or Fourier Trigonometric expansion corresponding to \( f \).

**Note:** Each term of the series is periodic with period 2\( L \) in this case we say that the series represents the periodic extension of \( f \) \( \forall x \).

A popular form of the Fourier series is obtained when \( L = \pi \) then the series is
\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad -\pi < x < \pi
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.
\]

The formula for \( a_n, b_n \) are known as the Euler formula for the series.

**Examples:**

1) Find the Fourier series of the periodic function
\[
f(x) = \begin{cases} 
0, & \text{if} \quad -2 < x < 0; \\
1, & \text{if} \quad 0 < x < 2. 
\end{cases}
\]

2) Find the Fourier series of \( f(x) = x^2, \quad -1 \leq x \leq 1 \) and that has period 2.
Theorem 3.2.1 A Fourier Convergence theorem:
Let \( f \) be periodic function with period 2L and that PWS on \([-L, L]\)

Then the corresponding Fourier Series \( \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\pi x/L) + b_n \sin(n\pi x)/L] \), \(-L < x < L\)

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(n\pi x/L) \, dx, \quad n \in \mathbb{N}_0
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(n\pi x/L) \, dx, \quad n \in \mathbb{N}.
\]

Converges to the average \( \frac{f(x^+) + f(x^-)}{2} \), \( x \) is a point of discontinuity.

Note: Fourier series converge to \( f(x) \) if \( f \) is continuous at \( x \).

Examples:
1) Find the convergence of the F.S. for \( f(x) = x^2 \) at \( x = 1 \) in the last example.
2) Find the convergence of the F.S. for \( f(x) = \begin{cases} 2, & \text{if } -1 < x < 0; \\ 1, & \text{if } 0 < x < 1. \end{cases} \)

at \( x = 0, 1, 7, 1/2, -1/2, 5/2. \)
3.3 Even and odd functions

\( f \) is even if \( f(-x) = f(x) \).

\( f \) is odd if \( f(-x) = -f(x) \).

If \( f, g \) both even then \( fg \) is even.

If \( f \) is even and \( g \) is odd then \( fg \) is odd.

If \( f, g \) both odd then \( fg \) is even.

If \( f \) is even then
\[
\int_{-L}^{L} f(x)\,dx = 2 \int_{0}^{L} f(x)\,dx.
\]

If \( f \) is odd then
\[
\int_{-L}^{L} f(x)\,dx = 0.
\]

Examples: Test the functions for even or odd.
\( x^2, \cos x, \sin x, x^3, e^x, e^x + e^{-x}, e^x - e^{-x}, x^3 + x^2, 1, 0. \)

3.4 Fourier Sine and Cosine series

**First:** If \( f \) is even:

Then \( f(x) \cos(n\pi x/L) \) is even and
\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(n\pi x/L)\,dx = \frac{2}{L} \int_{0}^{L} f(x) \cos(n\pi x/L)\,dx.
\]

and \( f(x) \sin(n\pi x/L) \) is odd then:
\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(n\pi x/L)\,dx = 0.
\]

Then the Fourier Series is
\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L),
\]
\[
a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos(n\pi x/L)\,dx.
\]

The interval in this case is \((0, L)\) but the even periodic extension of \( f \) give a period of \( 2L \).
This series is called Fourier Cosine series or the half range Fourier cosine series.

**Second:** If \( f \) is odd:

Then \( f(x) \sin(n\pi x/L) \) is even and
\[
b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin(n\pi x/L)\,dx
\]

and \( f(x) \cos(n\pi x/L) \) is odd and \( a_n = 0 \).

Then the Fourier Series is
\[ f(x) \sim \sum_{n=1}^{\infty} b_n \sin(n\pi x/L), \]
\[ b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin(n\pi x/L) \, dx. \]

Again the interval is \((0, L)\) but the period is \(2L\).
The series is called Fourier sine series.

Examples:

1) a) Find the Fourier series for the function:
\[ f(x) = \begin{cases} 
-\cos x, & \text{if } -\pi < x < 0; \\
\cos x, & \text{if } 0 < x < \pi .
\end{cases} \]

   b) Find the convergence at all jumps discontinuities.

2) a) Find the Fourier series for \( f(x) = 4 - x^2, \quad -2 \leq x \leq 2. \)

   b) Show that \( \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}. \)
3.5 Finite Fourier Transforms

Definition 3.5.1: Let \( f \) be PWC function on \((0, L)\)

1) The finite Fourier sine transform of \( f \) is

\[
S_n\{f\} = \int_0^L f(x) \sin(n\pi x/L) \, dx = F_s(n), \quad n \in \mathbb{N}.
\]

The inverse of the transform is the Fourier series with the factor \(2/L\).

\[
f(x) = \frac{2}{L} \sum_{n=1}^{\infty} F_s(n) \sin(n\pi x/L).
\]

2) The finite Fourier cosine transform of \( f \) is

\[
C_n\{f\} = \int_0^L f(x) \cos(n\pi x/L) \, dx = F_c(n), \quad n \in \mathbb{N}_0.
\]

where the inverse is

\[
f(x) = \frac{F_c(0)}{L} + \frac{2}{L} \sum_{n=1}^{\infty} F_c(n) \cos(n\pi x/L).
\]

Examples:

1) If \( f' \) is continuous and \( f'' \) is PWC on \([0, L]\) show that

\[
S_n\{f''\} = \frac{n\pi}{L} [f(0) - (-1)^n f(L)] - \frac{n^2\pi^2}{L^2} F_s(n).
\]

2) Find \( C_n\{1\}, \quad C_n\{e^{kx}\}, \quad S_n\{x\}. \)