Integral Relations for a Control Volume

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Introduction

• In analysing fluid motion, we might take one of two paths:
  1. Seeking to describe the detailed flow pattern at every point \((x, y, z)\) in the field OR
  2. working with a finite region, making a balance of flow in versus flow out, and determining gross flow effects such as the force or torque on a body or the total energy exchange

• The second is the “control-volume” method and is the subject of this chapter.

• The first is the “differential” approach and is developed in Chap. 4.

• Control volume is a very quick analysis that give a quantitative results.
  • In Fluids: control volume used for conservation of mass, linear momentum, angular momentum, and energy
  • In thermodynamics it is mainly conservation of mass and energy.
What is a control volume?

- **control volume approach**: is to consider a fixed interior volume of a system.

- **System approach**: We follow the fluid as it moves and deforms.—no mass crosses the boundary, and the total mass of the system remains fixed.
  - an arbitrary quantity of mass of fixed identity. Everything external to this system is denoted by the term surroundings,
• Control volumes can be Fixed, moving, and deformable

a) Fixed control volume for nozzle-stress analysis;
b) Control volume moving at ship speed for drag-force analysis;
c) Control volume deforming within cylinder for transient pressure-variation analysis.
Mass and Volume Flow Rates

**Mass flow rate:** The amount of mass flowing through a cross section per unit time.

The differential mass flow rate

\[
\delta m = \rho V_n \, dA_c
\]

**Point functions have exact differentials**

\[
\int_1^2 dA_c = A_{c2} - A_{c1} = \pi (r_2^2 - r_1^2)
\]

**Path functions have inexact differentials**

\[
\int_1^2 \delta m = \dot{m}_{\text{total}} \quad \text{not} \quad \dot{m}_2 - \dot{m}_1
\]

The normal velocity \( V_n \) for a surface is the component of velocity perpendicular to the surface.
\[ \delta m = \rho V_n \, dA_c \]

\[ \dot{m} = \int_{A_c} \delta m = \int_{A_c} \rho V_n \, dA_c \]

**Mass flow rate**

\[ \dot{m} = \rho \bar{V} = \frac{\dot{V}}{V_c} \]

**Average velocity**

\[ V_{\text{avg}} = \frac{1}{A_c} \int_{A_c} V_n \, dA_c \]

**Volume flow rate**

\[ \dot{V} = \int_{A_c} V_n \, dA_c = V_{\text{avg}} A_c = VA_c \quad (\text{m}^3/\text{s}) \]

The average velocity \( V_{\text{avg}} \) is defined as the average speed through a cross section.

The volume flow rate is the volume of fluid flowing through a cross section per unit time.
EXAMPLE 3.4

For steady viscous flow through a circular tube (Fig. E3.4), the axial velocity profile is given approximately by

\[ u = U_0 \left( 1 - \frac{r}{R} \right)^m \]

so that \( u \) varies from zero at the wall \( (r = R) \), or no slip, up to a maximum \( u = U_0 \) at the centerline \( r = 0 \). For highly viscous (laminar) flow \( m \approx \frac{1}{2} \), while for less viscous (turbulent) flow \( m \approx \frac{1}{7} \). Compute the average velocity if the density is constant.

Solution

The average velocity is defined by Eq. (3.32). Here \( \mathbf{V} = u \mathbf{i} \) and \( \mathbf{n} = \mathbf{i} \), and thus \( \mathbf{V} \cdot \mathbf{n} = u \). Since the flow is symmetric, the differential area can be taken as a circular strip \( dA = 2 \pi r \, dr \). Equation (3.32) becomes

\[ V_{av} = \frac{1}{A} \int u \, dA = \frac{1}{\pi R^2} \int_0^R U_0 \left( 1 - \frac{r}{R} \right)^m 2\pi r \, dr \]

or

\[ V_{av} = U_0 \frac{2}{(1 + m)(2 + m)} \quad \text{Ans.} \]

For the laminar-flow approximation, \( m \approx \frac{1}{2} \) and \( V_{av} \approx 0.53 U_0 \). (The exact laminar theory in Chap. 6 gives \( V_{av} = 0.50 U_0 \).) For turbulent flow, \( m \approx \frac{1}{7} \) and \( V_{av} \approx 0.82 U_0 \). (There is no exact turbulent theory, and so we accept this approximation.) The turbulent velocity profile is more uniform across the section, and thus the average velocity is only slightly less than maximum.
Water, assumed incompressible, flows steadily through the round pipe in Fig. P3.15. The entrance velocity is constant, \( u = U_0 \), and the exit velocity approximates turbulent flow, \( u = u_{\max}(1 - r/R)^{1/7} \). Determine the ratio \( U_0/u_{\max} \) for this flow.

\[ Q_1 = Q_2, \quad \text{or:} \quad \int_0^R U_0 2\pi r \, dr = \int_0^R u_{\max} \left(1 - \frac{r}{R}\right)^{1/7} 2\pi r \, dr, \]

\[ U_o \pi R^2 = u_{\max} \frac{49\pi}{60} R^2 \]

\[ \frac{U_o}{u_{\max}} = \frac{49}{60} \]
Review, Basic laws for a fixed mass or system approach

- **Conservation of mass:** 
  \[
  \frac{dm}{dt} = 0
  \]
  Mass of the system can be defined as:
  \[
  M = \int dm = \int_{V(\text{system})} \rho dV
  \]

- **Conservation of Linear momentum, \(p\):** 
  \[
  p = mv
  \]
  \[
  F = \frac{dp}{dt} = \dot{m}v = ma \quad \text{(time rate of change of momentum)}
  \]
  Linear momentum: 
  \[
  p = \int_{(\text{sys})} \dot{v} \, dm = \int_{V(\text{sys})} \dot{v} \, \rho dV
  \]
• Conservation of Energy

\[ \delta Q + \delta W = dE \]

Where \( Q \) is heat and \( W \) is work

\[ \dot{Q} + \dot{W} = \left. \frac{dE}{dt} \right|_{\text{system}} \]

\[ E_{\text{system}} = \int_{\text{system}} e \cdot dm = \int_{\text{system}} e \cdot \rho dV \]

Where \( e = u + \frac{v^2}{2} + gz \).

• The main aim of the following sections is translate from \textbf{fixed mass state} where these equations were developed for into \textbf{control volume} where we have the mass crossing the boundaries.
Review, intensive and extensive properties

• Extensive properties = $B$ (total mass, Linear momentum and energy)
• Intensive properties = $b = B/m$

<table>
<thead>
<tr>
<th></th>
<th>Extensive property</th>
<th>Intensive property</th>
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<tr>
<td>Conservation of mass</td>
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<td>1</td>
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<tr>
<td>Conservation of momentum</td>
<td>$p = mv$</td>
<td>$v$</td>
</tr>
<tr>
<td>Conservation of Energy</td>
<td>$E$</td>
<td>$e$</td>
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Review. Classification of fluid

Fluid flow may be classified as:

• **Uniform Flow conditions** (velocity, pressure, cross section or depth) are the same at every point in the fluid.

• **Non-uniform Flow**: Flow conditions are not the same at every point.

• **Steady Flow**: Flow conditions could differ from point to point but it doesn’t change with time.

• **Unsteady Flow**: If at any point in the fluid, the conditions change with time, the flow is described as unsteady.
THE REYNOLDS TRANSPORT THEOREM

• The previous equations in the system approach are all expressed in time rate of change for \( \frac{dB}{dt} \)_{system}.

• We need now to relate the change in extensive property B for a system to a control volume.

• The relationship between the time rates of change of an extensive property for a System, B and for a control volume is expressed by the Reynolds transport theorem (RTT).

The Reynolds transport theorem (RTT) provides a link between the system approach and the control volume approach.
The time rate of change of the extensive property $B$ of the system is **EQUAL** to the time rate of change of extensive $B$ of the control volume PLUS Net rate of flux of extensive property $B$ through the control surface.

For a Uniform flow:

\[
\frac{d B_{sys}}{dt} = \frac{d B_{CV}}{dt} - \dot{B}_{in} + \dot{B}_{out}
\]

Where

\[
B_{CV} = \int_{CV} \rho b \, dV
\]

&

\[
\dot{B}_{net} = \dot{B}_{out} - \dot{B}_{in} = \int_{CS} \rho b \vec{V} \cdot \vec{n} \, dA
\]
Outflow and inflow of mass across the differential area of a control surface.
RTT Special Cases

• For steady flow, the time derivative drops out,

\[ \frac{dB_{sys}}{dt} = \frac{d}{dt} \int_{CV} \rho b \, dV + \int_{CS} \rho b \vec{V} \cdot \vec{n} \, dA \]

\[\text{RTT, steady flow:} \quad \frac{dB_{sys}}{dt} = \int_{CS} \rho b \vec{V}_r \cdot \vec{n} \, dA\]
RTT Special Cases

For a uniform velocity control volumes with well-defined inlets and outlets

Approximate RTT for well-defined inlets and outlets:

\[
\frac{d B_{sys}}{dt} = \frac{d}{dt} \int_{CV} \rho b dV + \sum_{\text{out}} \frac{m_r}{b_{avg}} - \sum_{\text{in}} \frac{m_r}{b_{avg}}
\]

for each outlet

for each inlet

\[
\dot{m}_r \approx \rho_{avg} \dot{V}_r = \rho_{avg} V_{r,avg} A
\]
RTT Special Cases, 
For **moving** and/or **deforming** control volumes

- For a control volume moving at constant velocity
- The absolute velocity $V$ in the second term is replaced by the **relative velocity**

\[
\vec{V}_r = \vec{V} - \vec{V}_{CS}
\]

**Relative velocity:**

**RTT, steady flow:**

\[
\frac{dB_{sys}}{dt} = \int_{CS} \rho b \vec{V}_r \cdot \vec{n} \, dA
\]

**RTT, nonfixed CV:**

\[
\frac{dB_{sys}}{dt} = \frac{d}{dt} \int_{CV} \rho b \, dV + \int_{CS} \rho b \vec{V}_r \cdot \vec{n} \, dA
\]

Reynolds transport theorem applied to a control volume moving at constant velocity.
Conservation of Mass Principle

The conservation of mass equation is obtained by replacing $B$ in the Reynolds transport theorem by mass $m$, and $b$ by 1 ($m$ per unit mass $= m/m = 1$).

General conservation of mass:

$$\frac{d}{dt} \int_{CV} \rho \, d\mathbf{V} + \sum_{\text{out}} \rho |V_n| A - \sum_{\text{in}} \rho |V_n| A = 0$$

$$\frac{d}{dt} \int_{CV} \rho \, d\mathbf{V} = \sum_{\text{in}} \dot{m} - \sum_{\text{out}} \dot{m}$$

$$\frac{dm_{CV}}{dt} = \sum_{\text{in}} \dot{m} - \sum_{\text{out}} \dot{m}$$

(a) Control surface at an angle to the flow

$$\dot{m} = \rho V A$$

(b) Control surface normal to the flow

$$\dot{m} = \rho V A$$
Steady—Flow Processes

For steady flow, the total amount of mass contained in CV is constant.

For incompressible flows,

\[ \sum_{in} V_n A_n = \sum_{out} V_n A_n \]
Special Case: Incompressible Flow

The conservation of mass relations can be simplified even further when the fluid is incompressible, which is usually the case for liquids.

\[ \dot{m}_2 = 2 \text{ kg/s} \]
\[ \dot{V}_2 = 0.8 \text{ m}^3/\text{s} \]

\[ \sum_{\text{in}} \dot{V} = \sum_{\text{out}} \dot{V} \quad (\text{m}^3/\text{s}) \]

Steady, incompressible

\[ \dot{V}_1 = \dot{V}_2 \rightarrow V_1A_1 = V_2A_2 \]

Steady, incompressible flow (single stream)

There is no such thing as a “conservation of volume” principle.

However, for steady flow of liquids, the volume flow rates, as well as the mass flow rates, remain constant since liquids are essentially incompressible substances.

During a steady-flow process, volume flow rates are not necessarily conserved although mass flow rates are.
EXAMPLE 3.3

Write the conservation-of-mass relation for steady flow through a streamtube (flow everywhere parallel to the walls) with a single one-dimensional exit 1 and inlet 2 (Fig. E3.3).

Solution

For steady flow Eq. (3.24) applies with the single inlet and exit

\[ \dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \text{const} \]

Thus, in a streamtube in steady flow, the mass flow is constant across every section of the tube. If the density is constant, then

\[ Q = A_1 V_1 = A_2 V_2 = \text{const} \quad \text{or} \quad V_2 = \frac{A_1}{A_2} V_1 \]
EXAMPLE 3.6

The tank in Fig. E3.6 is being filled with water by two one-dimensional inlets. Air is trapped at the top of the tank. The water height is \( h \). (a) Find an expression for the change in water height \( dh/dt \). (b) Compute \( dh/dt \) if \( D_1 = 1 \text{ in} \), \( D_2 = 3 \text{ in} \), \( V_1 = 3 \text{ ft/s} \), \( V_2 = 2 \text{ ft/s} \), and \( A_t = 2 \text{ ft}^2 \), assuming water at 20°C.

Part (a) A suggested control volume encircles the tank and cuts through the two inlets. The flow within is unsteady, and Eq. (3.22) applies with no outlets and two inlets:

\[
\frac{d}{dt} \left( \int_{CV} \rho \, dV \right) - \rho_1 A_1 V_1 - \rho_2 A_2 V_2 = 0
\]  
(1)

Now if \( A_t \) is the tank cross-sectional area, the unsteady term can be evaluated as follows:

\[
\frac{d}{dt} \left( \int_{CV} \rho \, dV \right) = \frac{d}{dt} \left( \rho_w A_t h \right) + \frac{d}{dt} \left[ \rho_a A_t (H - h) \right] = \rho_w A_t \frac{dh}{dt}
\]  
(2)

The \( \rho_a \) term vanishes because it is the rate of change of air mass and is zero because the air is trapped at the top. Substituting (2) into (1), we find the change of water height

\[
\frac{dh}{dt} = \frac{\rho_1 A_1 V_1 + \rho_2 A_2 V_2}{\rho_w A_t}
\]  
Ans. (a)

For water, \( \rho_1 = \rho_2 = \rho_w \), and this result reduces to

\[
\frac{dh}{dt} = \frac{A_1 V_1 + A_2 V_2}{A_t} = \frac{Q_1 + Q_2}{A_t}
\]  
(3)
Part (b) The two inlet volume flows are

\[ Q_1 = A_1 V_1 = \frac{1}{4} \pi \left( \frac{1}{12} \text{ ft} \right)^2 (3 \text{ ft/s}) = 0.016 \text{ ft}^3/\text{s} \]
\[ Q_2 = A_2 V_2 = \frac{1}{4} \pi \left( \frac{3}{12} \text{ ft} \right)^2 (2 \text{ ft/s}) = 0.098 \text{ ft}^3/\text{s} \]

Then, from Eq. (3),

\[ \frac{dh}{dt} = \frac{(0.016 + 0.098) \text{ ft}^3/\text{s}}{2 \text{ ft}^2} = 0.057 \text{ ft/s} \]

*Ans. (b)*

*Suggestion:* Repeat this problem with the top of the tank open.
conservation of Linear momentum

• In linear momentum the variable is $\mathbf{B} = m\mathbf{V}$ and intensive property $b = \frac{d\mathbf{B}}{dm} = \mathbf{V}$, and application of the Reynolds transport theorem gives the linear-momentum relation for a deformable control volume

$$\frac{d}{dt} (m\mathbf{V})_{\text{syst}} = \sum \mathbf{F} = \frac{d}{dt} \left( \int_{\text{CV}} \mathbf{V} \rho \, dV \right) + \int_{\text{CS}} \mathbf{V} \rho (\mathbf{V}_r \cdot \mathbf{n}) \, dA$$

The term $\sum \mathbf{F}$ is the vector sum of all forces acting on the control-volume material considered as a free body; i.e., it includes:
• surface forces $\mathbf{F}_s$ on all fluids: due to shear force and pressure.
• Body force $\mathbf{F}_b$ (gravity and electromagnetic) acting on the masses within the control volume.

The velocity is the fluid velocity relative to an inertial (nonaccelerating) coordinate system;
Linear momentum special case

• In general in RTT equation, a uniform velocity control volumes with well-defined inlets and outlets will be as follows

\[
\frac{d B_{sys}}{dt} = \frac{d}{dt} \int_{CV} \rho b \, d\mathcal{V} + \sum_{\text{out}} \dot{m}_r b_{avg} - \sum_{\text{in}} \dot{m}_r b_{avg}
\]

• Applying this to Linear momentum

\[
\sum F = \frac{d}{dt} \left( \int_{CV} \rho v \, d\mathcal{V} \right) + \sum (\dot{m}_i V_i)_{\text{out}} - \sum (\dot{m}_i V_i)_{\text{in}}
\]
Net pressure force on a close control volume

\[ F_{\text{press}} = \int_{CS} (p - p_a)(-\mathbf{n}) \, dA = \int_{CS} p_{gage}(-\mathbf{n}) \, dA \]

- Pressure Inwards is positive
- Pressure outwards is negative
As shown in Fig. 3.9a, a fixed vane turns a water jet of area $A$ through an angle $\theta$ without changing its velocity magnitude. The flow is steady, pressure is $p_a$ everywhere, and friction on the vane is negligible. (a) Find the components $F_x$ and $F_y$ of the applied vane force. (b) Find expressions for the force magnitude $F$ and the angle $\phi$ between $\mathbf{F}$ and the horizontal; plot them versus $\theta$.

\[
F_x = mV(\cos \theta - 1) \quad F_y = mV \sin \theta
\]

\[
F = (F_x^2 + F_y^2)^{1/2} = mV[\sin^2 \theta + (\cos \theta - 1)^2]^{1/2} = 2mV \sin \frac{\theta}{2}
\]

But the magnitude $V_1 = V_2 = V$ as given, and conservation of mass for the streamtube requires $\dot{m}_1 = \dot{m}_2 = \dot{m} = \rho AV$. The vector diagram for force and momentum change becomes an isosce-
A water jet of velocity $V_j$ impinges normal to a flat plate which moves to the right at velocity $V_c$, as shown in Fig. 3.10a. Find the force required to keep the plate moving at constant velocity if the jet density is 1000 kg/m$^3$, the jet area is 3 cm$^2$, and $V_j$ and $V_c$ are 20 and 15 m/s, respectively. Neglect the weight of the jet and plate, and assume steady flow with respect to the moving plate with the jet splitting into an equal upward and downward half-jet.

1. $A_1 = \frac{1}{2} A_j$
2. $A_2 = \frac{1}{2} A_j$
\[ \dot{m}_{\text{out}} = \dot{m}_{\text{in}} \]

\[ \rho_1 A_1 V_1 + \rho_2 A_2 V_2 = \rho_j A_j (V_j - V_c) \]

\[ V_1 + V_2 = 2(V_j - V_c) \]

\[ V_1 = V_2 = 20 - 15 = 5 \text{ m/s} \]

\[ R_x = -\dot{m}_j u_j = -[\rho_j A_j (V_j - V_c)](V_j - V_c) \]

\[ R_x = -(1000 \text{ kg/m}^3)(0.0003 \text{ m}^2)(5 \text{ m/s})^2 = -7.5 \text{ (kg} \cdot \text{m)/s}^2 = -7.5 \text{ N} \]
Momentum-Flux Correction Factor

- For flow in a duct, the axial velocity is usually nonuniform,
- For this case the simple momentum-flux calculation $\int u \rho (V \cdot n) dA = \dot{m}V = \rho AV^2$ is somewhat in error and should be corrected to $\beta \rho AV^2$. Where $\beta$ is the dimensionless momentum-flux correction factor, $\beta > 1$.

\[ \rho \int u^2 dA = \beta \dot{m} V_{av} = \beta \rho AV^2 \]

or

\[ \beta = \frac{1}{A} \int \left( \frac{u}{V_{av}} \right)^2 dA \]

Values of $\beta$ can be computed based on typical duct velocity profiles similar to those in Example 3.4. The results are as follows:

- Laminar flow: $u = U_0 \left( 1 - \frac{r^2}{R^2} \right)$, $\beta = \frac{4}{3}$ (3.43b)
- Turbulent flow: $u \approx U_0 \left( 1 - \frac{r}{R} \right)^m$, $\frac{1}{9} \leq m \leq \frac{1}{5}$

\[ \beta = \frac{(1 + m)^2(2 + m)^2}{2(1 + 2m)(2 + 2m)} \] (3.43c)

The turbulent correction factors have the following range of values:

<table>
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<tr>
<th>$m$</th>
<th>$\frac{1}{3}$</th>
<th>$\frac{1}{4}$</th>
<th>$\frac{1}{5}$</th>
<th>$\frac{1}{6}$</th>
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<tr>
<td>$\beta$</td>
<td>1.037</td>
<td>1.027</td>
<td>1.020</td>
<td>1.016</td>
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</table>
Noninertial Reference Frame

• All previous derivations and examples in this section have assumed that the coordinate system is inertial, i.e., at rest or moving at constant velocity.

• In many cases it is convenient to use a noninertial, or accelerating, coordinate system. An example would be coordinates fixed to a rocket during takeoff.

• Suppose that the fluid flow has velocity $V$ relative to a noninertial $xyz$ coordinate system, as shown in. Then $dV/dt$ will represent a noninertial acceleration which must be added vectorially to a relative acceleration $a_{\text{rel}}$ to give the absolute acceleration $a_i$ relative to some inertial coordinate system $XYZ$, a

$$a_i = \frac{dV}{dt} + a_{\text{rel}}$$
The RTT equation will be modified to account for the non-inertial reference

\[ \sum F - \int_{CV} a_{rel} dm = \frac{d}{dt} \left( \int_{CV} v_p dV \right) + \int_{CS} v(v.n) dA \]
• Assume the rocket start from rest and ignore drage
• Exhaust flow rate: $\dot{m}_e = 5\text{kg/s}$
• Exhaust Velocity $v_e=3500\text{m/s}$
• $m_0=400\text{ kg}$
• Find
  a) Initial acceleration
  b) $U(t)$ or velocity as a function of time
Questions to answer

• Assuming friction has a very little effect

\[ Q_{\text{net in}} + W_{\text{shaft, net in}} + W_{\text{pressure, net in}} = \frac{dE_{\text{sys}}}{dt} \]

\[ e = u + ke + pe = u + \frac{V^2}{2} + gz \]

\[ \frac{dE_{\text{sys}}}{dt} = \frac{d}{dt} \int_{CV} e \rho \, dV + \int_{CS} e \rho (\vec{V}_r \cdot \hat{n}) A \]

\[ Q_{\text{net in}} + W_{\text{shaft, net in}} + W_{\text{pressure, net in}} = \frac{d}{dt} \int_{CV} e \rho \, dV + \int_{CS} e \rho (\vec{V}_r \cdot \hat{n}) \, dA \]

(The net rate of energy transfer into a CV by heat and work transfer) = (The time rate of change of the energy content of the CV) + (The net flow rate of energy out of the control surface by mass flow)

\[ Q_{\text{net in}} + W_{\text{shaft, net in}} = \frac{d}{dt} \int_{CV} e \rho \, dV + \int_{CS} \left( \frac{P}{\rho} + e \right) \rho (\vec{V}_r \cdot \hat{n}) \, dA \]
In a typical engineering problem, the control volume may contain many inlets and outlets; energy flows in at each inlet, and energy flows out at each outlet. Energy also enters the control volume through net heat transfer and net shaft work.

\[
\dot{m} = \int_{A_c} \rho (\vec{V} \cdot \hat{n}) \, dA_c
\]

\[
\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} = \frac{d}{dt} \int_{CV} e \rho \, d\mathcal{V} + \sum_{\text{out}} \dot{m} \left( \frac{P}{\rho} + e \right) - \sum_{\text{in}} \dot{m} \left( \frac{P}{\rho} + e \right) = u + V^2/2 + gz
\]

\[
\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} = \frac{d}{dt} \int_{CV} e \rho \, d\mathcal{V} + \sum_{\text{out}} \dot{m} \left( \frac{P}{\rho} + u + \frac{V^2}{2} + gz \right) - \sum_{\text{in}} \dot{m} \left( \frac{P}{\rho} + u + \frac{V^2}{2} + gz \right)
\]

\[
\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} = \frac{d}{dt} \int_{CV} e \rho \, d\mathcal{V} + \sum_{\text{out}} \dot{m} \left( h + \frac{V^2}{2} + gz \right) - \sum_{\text{in}} \dot{m} \left( h + \frac{V^2}{2} + gz \right)
\]

\[
h = u + P\nu = u + P/\rho.
\]
ENERGY ANALYSIS OF STEADY FLOWS

The net rate of energy transfer to a control volume by heat transfer and work during steady flow is equal to the difference between the rates of outgoing and incoming energy flows by mass flow.

\[ \dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} = \sum_{\text{out}} \dot{m} \left( h + \frac{V^2}{2} + gz \right) - \sum_{\text{in}} \dot{m} \left( h + \frac{V^2}{2} + gz \right) \]

single-stream devices

\[ q_{\text{net in}} + w_{\text{shaft, net in}} = h_2 - h_1 + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) \]

\[ h = u + PV = u + P/\rho \]

\[ w_{\text{shaft, net in}} + \frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 + (u_2 - u_1 - q_{\text{net in}}) \]

A control volume with only one inlet and one outlet and energy interactions.
Ideal flow (no mechanical energy loss):

\[ q_{\text{net in}} = u_2 - u_1 \]

Real flow (with mechanical energy loss):

\[ e_{\text{mech, loss}} = u_2 - u_1 - q_{\text{net in}} \]

\[ e_{\text{mech, in}} = e_{\text{mech, out}} + e_{\text{mech, loss}} \]

\[ w_{\text{shaft, net in}} + \frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 + e_{\text{mech, loss}} \]

\[ w_{\text{shaft, net in}} = w_{\text{pump}} - w_{\text{turbine}} \]

\[ \frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 + w_{\text{pump}} = \frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 + w_{\text{turbine}} + e_{\text{mech, loss}} \]

\[ m \left( \frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 \right) + \dot{W}_{\text{pump}} = m \left( \frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 \right) + \dot{W}_{\text{turbine}} + \dot{E}_{\text{mech, loss}} \]

\[ \dot{E}_{\text{mech, loss}} = \dot{E}_{\text{mech loss, pump}} + \dot{E}_{\text{mech loss, turbine}} + \dot{E}_{\text{mech loss, piping}} \]
A typical power plant has numerous pipes, elbows, valves, pumps, and turbines, all of which have irreversible losses.
Energy equation in terms of heads

\[ \frac{P_1}{\rho_1 g} + \frac{V_1^2}{2g} + z_1 + h_{\text{pump, } u} = \frac{P_2}{\rho_2 g} + \frac{V_2^2}{2g} + z_2 + h_{\text{turbine, } e} + h_L \]

where

- \( h_{\text{pump, } u} = \frac{W_{\text{pump, } u}}{g} = \frac{\dot{W}_{\text{pump}}}{\dot{m}g} = \frac{\eta_{\text{pump}} \dot{W}_{\text{pump}}}{\dot{m}g} \) is the useful head delivered to the fluid by the pump. Because of irreversible losses in the pump, \( h_{\text{pump, } u} \) is less than \( \dot{W}_{\text{pump}}/\dot{m}g \) by the factor \( \eta_{\text{pump}} \).

- \( h_{\text{turbine, } e} = \frac{W_{\text{turbine, } e}}{g} = \frac{\dot{W}_{\text{turbine}}}{\dot{m}g} = \frac{\dot{W}_{\text{turbine}}}{\eta_{\text{turbine}} \dot{m}g} \) is the extracted head removed from the fluid by the turbine. Because of irreversible losses in the turbine, \( h_{\text{turbine, } e} \) is greater than \( \dot{W}_{\text{turbine}}/\dot{m}g \) by the factor \( \eta_{\text{turbine}} \).

- \( h_L = \frac{e_{\text{mech loss, piping}}}{g} = \frac{\dot{E}_{\text{mech loss, piping}}}{\dot{m}g} \) is the irreversible head loss between 1 and 2 due to all components of the piping system other than the pump or turbine.

Note, \( h_L \) is refer to as \( h_f \) in the Frank white book. The main reason for mechanical loss is friction loss that
Mechanical energy flow chart for a fluid flow system that involves a pump and a turbine. Vertical dimensions show each energy term expressed as an equivalent column height of fluid, i.e., head.
EXAMPLE 3.17

Gasoline at 20°C is pumped through a smooth 12-cm-diameter pipe 10 km long, at a flow rate of 75 m³/h (330 gal/min). The inlet is fed by a pump at an absolute pressure of 24 atm. The exit is at standard atmospheric pressure and is 150 m higher. Estimate the frictional head loss \( h_f \), and compare it to the velocity head of the flow \( V^2/(2g) \). (These numbers are quite realistic for liquid flow through long pipelines.)

For gasoline at 20°C, from Table A.3, \( \rho = 680 \text{ kg/m}^3 \), or \( \gamma = (680)(9.81) = 6670 \text{ N/m}^3 \).

\[
\frac{P_{in}}{\gamma} + \frac{V_{in}^2}{2g} + z_{in} = \frac{P_{out}}{\gamma} + \frac{V_{out}^2}{2g} + z_{out} + h_f
\]

The pipe is of uniform cross section, and thus the average velocity everywhere is

\[
V_{in} = V_{out} = \frac{Q}{A} = \frac{(75/3600) \text{ m}^3/\text{s}}{\pi/4(0.12 \text{ m})^2} = 1.84 \text{ m/s}
\]

Being equal at inlet and exit, this term will cancel out of Eq. (1) above, but we are asked to compute the velocity head of the flow for comparison purposes:

\[
\frac{V^2}{2g} = \frac{(1.84 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} = 0.173 \text{ m}
\]

Now we are in a position to evaluate all terms in Eq. (1) except the friction head loss:

\[
\frac{(24)(101,350 \text{ N/m}^2)}{6670 \text{ N/m}^3} + 0.173 \text{ m} + 0 \text{ m} = \frac{101,350 \text{ N/m}^2}{6670 \text{ N/m}^3} + 0.173 \text{ m} + 150 \text{ m} + h_f
\]

or

\[ h_f = 364.7 - 15.2 - 150 \approx 199 \text{ m} \]
P3.130 When the pump in Fig. P3.130 draws 220 m$^3$/h of water at 20°C from the reservoir, the total friction head loss is 5 m. The flow discharges through a nozzle to the atmosphere. Estimate the pump power in kW delivered to the water.
\[ V_2 = \frac{Q}{A_2} = \frac{220/3600}{\pi(0.025)^2} = 31.12 \text{ m/s}, \quad \text{while} \quad V_1 \approx 0 \text{ (reservoir surface)} \]

Now apply the steady flow energy equation from (1) to (2):

\[ \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_f - h_p, \]

or:  \[ 0 + 0 + 0 = 0 + \frac{(31.12)^2}{2(9.81)} + 2 + 5 - h_p, \quad \text{solve for} \quad h_p \approx 56.4 \text{ m}. \]

The pump power \[ P = \rho g Q h_p = (998)(9.81)(220/3600)(56.4) \]

\[ = 33700 \text{ W} = \boxed{33.7 \text{ kW}} \quad \text{Ans.} \]
P3.126 There is a steady isothermal flow of water at 20°C through the device in Fig. P3.126. Heat-transfer, gravity, and temperature effects are negligible. Known data are $D_1 = 9$ cm, $Q_1 = 220$ m$^3$/h, $p_1 = 150$ kPa, $D_2 = 7$ cm, $Q_2 = 100$ m$^3$/h, $p_2 = 225$ kPa, $D_3 = 4$ cm, and $p_3 = 265$ kPa. Compute the rate of shaft work done for this device and its direction.
Solution: For continuity, \( Q_3 = Q_1 - Q_2 = 120 \text{ m}^3/\text{hr} \). Establish the velocities at each port:

\[
V_1 = \frac{Q_1}{A_1} = \frac{220}{3600} \pi (0.045)^2 = 9.61 \text{ m/s} \quad V_2 = \frac{100}{3600} \pi (0.035)^2 = 7.22 \text{ m/s} \quad V_3 = \frac{120}{3600} \pi (0.02)^2 = 26.5 \text{ m/s}
\]

With gravity and heat transfer and internal energy neglected, the energy equation becomes

\[
\dot{Q} - \dot{W}_s - \dot{W}_v = \dot{m}_3 \left( \frac{p_3}{\rho_3} + \frac{V_3^2}{2} \right) + \dot{m}_2 \left( \frac{p_2}{\rho_2} + \frac{V_2^2}{2} \right) - \dot{m}_1 \left( \frac{p_1}{\rho_1} + \frac{V_1^2}{2} \right),
\]

or:

\[
\begin{align*}
\dot{W}_s/\rho &= \frac{100}{3600} \left[ \frac{225000}{998} + \frac{(7.22)^2}{2} \right] + \frac{120}{3600} \left[ \frac{265000}{998} + \frac{(26.5)^2}{2} \right] \\
&\quad + \frac{220}{3600} \left[ \frac{150000}{998} + \frac{(9.61)^2}{2} \right]
\end{align*}
\]

Solve for the shaft work:

\[
\dot{W}_s = 998(-6.99 - 20.56 + 12.00) \approx -15500 \text{ W} \quad \text{Ans.}
\]

(negative denotes work done \textit{on} the fluid)
THE BERNOUlli EQUATION

Bernoulli equation: An approximate relation between pressure, velocity, and elevation, and is valid in regions of steady, incompressible flow where net frictional forces are negligible.

Despite its simplicity, it has proven to be a very powerful tool in fluid mechanics.

The Bernoulli approximation is typically useful in flow regions outside of boundary layers and wakes, where the fluid motion is governed by the combined effects of pressure and gravity forces.

The Bernoulli equation is an approximate equation that is valid only in inviscid regions of flow where net viscous forces are negligibly small compared to inertial, gravitational, or pressure forces. Such regions occur outside of boundary layers and wakes.
Conditions for Bernoulli equations

1. **Steady flow** The Bernoulli equation is applicable to *steady flow*.
2. **Frictionless flow** Every flow involves some friction, no matter how small, and *frictional effects* may or may not be negligible.
3. **No shaft work** The Bernoulli equation is not applicable in a flow section that involves a pump, turbine, fan, or any other machine or impeller since such devices destroy the streamlines and carry out energy interactions with the fluid particles. When these devices exist, the energy equation should be used instead.
4. **Incompressible flow** Density is taken constant in the derivation of the Bernoulli equation. The flow is incompressible for liquids and also by gases at Mach numbers less than about 0.3.
5. **No heat transfer** The density of a gas is inversely proportional to temperature, and thus the Bernoulli equation should not be used for flow sections that involve significant temperature change such as heating or cooling sections.
6. **Flow along a streamline** Strictly speaking, the Bernoulli equation is applicable along a streamline. However, when a region of the flow is *irrotational* and there is negligibly small *vorticity* in the flow field, the Bernoulli equation becomes applicable *across* streamlines as well.
Derivation of the Bernoulli Equation

The forces acting on a fluid particle along a streamline.

\[ \sum F_s = m a_s = P \, dA - (P + dP) \, dA - W \sin \theta = mV \frac{dV}{ds} \]

\[ m = \rho V = \rho \, dA \, ds \quad W = mg = \rho g \, dA \, ds \]

\[ \sin \theta = \frac{dz}{ds} \quad -dP \, dA - \rho g \, dA \, ds \frac{dz}{ds} = \rho \, dA \, ds \, V \frac{dV}{ds} \]

\[ -dP - \rho g \, dz = \rho V \, dV \quad V \, dV = \frac{1}{2} \, d(V^2) \]

\[ \frac{dP}{\rho} + \frac{1}{2} \, d(V^2) + g \, dz = 0 \]

\[ \int \frac{dP}{\rho} + \frac{V^2}{2} + gz = \text{constant (along a streamline)} \]

Steady, incompressible flow:

\[ \frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant (along a streamline)} \]

The Bernoulli equation between any two points on the same streamline:

\[ \frac{P_1}{\rho} + \frac{V_{1}^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_{2}^2}{2} + gz_2 \]
The incompressible Bernoulli equation is derived assuming incompressible flow, and thus it should not be used for flows with significant compressibility effects.

The Bernoulli equation for unsteady, compressible flow:

\[
\int \frac{dP}{\rho} + \frac{\partial V}{\partial t} ds + \frac{V^2}{2} + gz = \text{constant}
\]
The Bernoulli equation states that the sum of the kinetic, potential, and flow energies of a fluid particle is constant along a streamline during steady flow.

\[
\frac{P}{\rho} + \frac{v^2}{2} + gz = \text{constant}
\]

- The Bernoulli equation can be viewed as the “conservation of mechanical energy principle.”
- This is equivalent to the general conservation of energy principle for systems that do not involve any conversion of mechanical energy and thermal energy to each other, and thus the mechanical energy and thermal energy are conserved separately.
- The Bernoulli equation states that during steady, incompressible flow with negligible friction, the various forms of mechanical energy are converted to each other, but their sum remains constant.
- There is no dissipation of mechanical energy during such flows since there is no friction that converts mechanical energy to sensible thermal (internal) energy.
- The Bernoulli equation is commonly used in practice since a variety of practical fluid flow problems can be analyzed to reasonable accuracy with it.
Static, Dynamic, and Stagnation Pressures

The kinetic and potential energies of the fluid can be converted to flow energy (and vice versa) during flow, causing the pressure to change. Multiplying the Bernoulli equation by the density gives

\[ P + \rho \frac{V^2}{2} + \rho gz = \text{constant (along a streamline)} \]

\( P \) is the static pressure: It does not incorporate any dynamic effects; it represents the actual thermodynamic pressure of the fluid. This is the same as the pressure used in thermodynamics and property tables.

\( \rho V^2/2 \) is the dynamic pressure: It represents the pressure rise when the fluid in motion is brought to a stop isentropically.

\( \rho gz \) is the hydrostatic pressure: It is not pressure in a real sense since its value depends on the reference level selected; it accounts for the elevation effects, i.e., fluid weight on pressure. (Be careful of the sign—unlike hydrostatic pressure \( \rho gh \) which increases with fluid depth \( h \), the hydrostatic pressure term \( \rho gz \) decreases with fluid depth.)

Total pressure: The sum of the static, dynamic, and hydrostatic pressures. Therefore, the Bernoulli equation states that the total pressure along a streamline is constant.
Example: Water Discharge from a Large Tank

Example: Spraying Water into the Air

\[
\frac{P_1}{\rho g} + \frac{V_1^2}{2g} \approx 0 \quad + \quad z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \quad \Rightarrow \quad z_1 = \frac{V_2^2}{2g}
\]

\[
\frac{P_1}{\rho g} + \frac{V_1^2}{2g} \approx 0 \quad + \quad z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \quad \Rightarrow \quad \frac{P_1}{\rho g} = \frac{P_{\text{atm}}}{\rho g} + z_2
\]
**Stagnation pressure**: The sum of the static and dynamic pressures. It represents the pressure at a point where the fluid is brought to a complete stop isentropically.

\[ P_{stag} = P + \rho \frac{V^2}{2} \text{ (kPa)} \]

\[ V = \sqrt{\frac{2(P_{stag} - P)}{\rho}} \]

Close-up of a **Pitot-static probe**, showing the stagnation pressure hole and two of the five static circumferential pressure holes.

The static, dynamic, and stagnation pressures measured using **piezometer tubes**.
Example: Velocity Measurement by a Pitot Tube

\[ \frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^0}{2g} + z_2 \]

\[ V_1 = \frac{P_2 - P_1}{\rho g} \]

\[ h_3 = 12 \text{ cm} \]
\[ h_2 = 7 \text{ cm} \]
\[ h_1 = 3 \text{ cm} \]

Water
\[ \bullet \quad 1 \]
\[ \bullet \quad 2 \]

Stagnation point

\[ P_1 = \rho g (h_1 + h_2) \]
\[ P_2 = \rho g (h_1 + h_2 + h_3) \]
Hydraulic Grade Line (HGL) and Energy Grade Line (EGL)

It is often convenient to represent the level of mechanical energy graphically using *heights* to facilitate visualization of the various terms of the Bernoulli equation. Dividing each term of the Bernoulli equation by $g$ gives

\[
\frac{P}{\rho g} + \frac{V^2}{2g} + z = H = \text{constant} \quad \text{(along a streamline)}
\]

*\(P/\rho g\) is the pressure head;* it represents the height of a fluid column that produces the static pressure $P$.

*\(V^2/2g\) is the velocity head;* it represents the elevation needed for a fluid to reach the velocity $V$ during frictionless free fall.

*\(z\) is the elevation head;* it represents the potential energy of the fluid.

An alternative form of the Bernoulli equation is expressed in terms of heads as: *The sum of the pressure, velocity, and elevation heads is constant along a streamline.*
**Hydraulic grade line (HGL),** \( P/\rho g + z \) The line that represents the sum of the static pressure and the elevation heads.

**Energy grade line (EGL),** \( P/\rho g + V^2/2g + z \) The line that represents the total head of the fluid.

**Dynamic head,** \( V^2/2g \) The difference between the heights of EGL and HGL.
In an idealized Bernoulli-type flow, EGL is horizontal and its height remains constant. But this is not the case for HGL when the flow velocity varies along the flow.

A steep jump occurs in EGL and HGL whenever mechanical energy is added to the fluid by a pump, and a steep drop occurs whenever mechanical energy is removed from the fluid by a turbine.
Notes on HGL and EGL

• For *stationary bodies* such as reservoirs or lakes, the EGL and HGL coincide with the free surface of the liquid.

• The EGL is always a distance $V^2/2g$ above the HGL. These two curves approach each other as the velocity decreases, and they diverge as the velocity increases.

• In an *idealized Bernoulli-type flow*, EGL is horizontal and its height remains constant.

• For *open-channel flow*, the HGL coincides with the free surface of the liquid, and the EGL is a distance $V^2/2g$ above the free surface.

• At a *pipe exit*, the pressure head is zero (atmospheric pressure) and thus the HGL coincides with the pipe outlet.

• The *mechanical energy loss* due to frictional effects (conversion to thermal energy) causes the EGL and HGL to slope downward in the direction of flow. The slope is a measure of the head loss in the pipe. A component, such as a valve, that generates significant frictional effects causes a sudden drop in both EGL and HGL at that location.

• A *steep jump/drop* occurs in EGL and HGL whenever mechanical energy is added or removed to or from the fluid (pump, turbine).

• The (gage) pressure of a fluid is zero at locations where the HGL *intersects* the fluid. The pressure in a flow section that lies above the HGL is negative, and the pressure in a section that lies below the HGL is positive.
Find a relation between nozzle discharge velocity $V^2$ and tank free-surface height $h$ as in Fig. E3.21. Assume steady frictionless flow.
Mass conservation is usually a vital part of Bernoulli analyses. If $A_1$ is the tank cross section and $A_2$ the nozzle area, this is approximately a one-dimensional flow with constant density, Eq. (3.30),

$$A_1 V_1 = A_2 V_2$$  

(1)

Bernoulli’s equation (3.77) gives

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$$

But since sections 1 and 2 are both exposed to atmospheric pressure $p_1 = p_2 = p_a$, the pressure terms cancel, leaving

$$V_2^2 - V_1^2 = 2g(z_1 - z_2) = 2gh$$  

(2)

Eliminating $V_1$ between Eqs. (1) and (2), we obtain the desired result:

$$V_2^2 = \frac{2gh}{1 - \frac{A_2^2}{A_1^2}}$$  

Ans. (3)

Generally the nozzle area $A_2$ is very much smaller than the tank area $A_1$, so that the ratio $A_2^2/A_1^2$ is doubly negligible, and an accurate approximation for the outlet velocity is

$$V_2 \approx (2gh)^{1/2}$$  

Ans. (4)
EXAMPLE 3.23

A constriction in a pipe will cause the velocity to rise and the pressure to fall at section 2 in the throat. The pressure difference is a measure of the flow rate through the pipe. The smoothly necked-down system shown in Fig. E3.23 is called a venturi tube. Find an expression for the mass flux in the tube as a function of the pressure change.
Bernoulli’s equation is assumed to hold along the center streamline

\[ \frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2 \]

If the tube is horizontal, \( z_1 = z_2 \) and we can solve for \( V_2 \):

\[ V_2^2 - V_1^2 = \frac{2 \Delta p}{\rho} \quad \Delta p = p_1 - p_2 \]  

(1)

We relate the velocities from the incompressible continuity relation

\[ A_1 V_1 = A_2 V_2 \]

or

\[ V_1 = \beta^2 V_2 \quad \beta = \frac{D_2}{D_1} \]  

(2)

Combining (1) and (2), we obtain a formula for the velocity in the throat

\[ V_2 = \left[ \frac{2 \Delta p}{\rho (1 - \beta^4)} \right]^{1/2} \]  

(3)

The mass flux is given by

\[ \dot{m} = \rho A_2 V_2 = A_2 \left( \frac{2 \rho \Delta p}{1 - \beta^4} \right)^{1/2} \]  

(4)