Maxwell’s Equations

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Review – Electrostatics and Magnetostatics

**Electrostatic Fields**
produced by stationary charges.

**Magnetostatic Fields**
produced by steady (DC) currents or stationary magnet materials.

Electrostatic Fields and Magnetostatic Fields do not vary with time (time invariant).

**Electromagnetic Fields**
produced by time-varying currents.
Maxwell’s Equations for *static* fields

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<td>$\vec{\nabla} \times \vec{H} = \vec{J}$</td>
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Michael Faraday's ideas about conservation of energy led him to believe that since an electric current could cause a magnetic field, a magnetic field should be able to produce an electric current. He demonstrated this principle of induction in 1831.
Faraday’s Law

- In 1831, Faraday discovered that a time-varying magnetic field would produce an induced voltage (called electromotive force or emf).
- According to Faraday’s experiments, a static magnetic field produces no current flow.

Faraday discovered that the induced Vemf (in volts) in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit.
Faraday’s Law

\[ V_{\text{emf}} = - \frac{d \lambda}{dt} = -N \frac{d \Psi}{dt} \]

where \( \lambda = N \Psi \) is the flux linkage
\( N \) is the number of turns in the circuit.
\( \Psi \) is the flux through each turn.
Faraday’s Law

Change flux due to moving permanent magnet

\[ V_{emf} = -N \frac{d\Psi}{dt} \]
Faraday’s Law
Induced EMF is in direction that oppose the change in flux that caused it.
9.3 Transformer and motional electromotive forces

For a circuit with a single turn, $N = 1$

$$V_{emf} = -\frac{d\Psi}{dt}$$

In terms of $E$ and $B$

$$V_{emf} = \oint_L E.dl = -\frac{d}{dt} \int_S B.dS$$

where $S$ is the surface area of the circuit bounded by the closed path $L$.

Notice that in time varying situation, both electric and magnetic fields are present and interrelated.
The variation of flux with time may be caused in three ways:

1) By having a stationary loop in a time-varying B field.
   
   *(Transformer induction)*

2) By having a time-varying loop area in static B field.

   *(Motional induction)*

3) By having a time-varying loop area in a time-varying B field.

   *(General case, transformer and motional induction)*
Application: DC Generators
Application: AC Generator

Water turns wheel
→ rotates magnet
→ changes flux
→ induces emf
→ drives current
A. Stationary loop in Time-Varying B field (Transformer EMF)

\[ \text{emf} = \int E \cdot dl = - \int_S \frac{\partial B}{\partial t} \cdot dS \]
A. Stationary loop in Time-Varying B field (Transformer EMF)

By applying Stoke's theorem

\[ \nabla \times E = -\frac{\partial B}{\partial t} \]

This is one of Maxwell’s equations for time-varying fields.

Note that \( \nabla \times E \neq 0 \) (time-varying E field is non-conservative)
B. Moving loop in static B field
(Motional EMF)

Consider a conducting loop moving with uniform velocity \( \mathbf{u} \), the emf induced in the loop is

\[
V_{\text{emf}} = \oint E_m \cdot \mathbf{dl} = \oint (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{dl}
\]

- This type of emf is called **motional emf** or **flux-cutting emf** (due to motion action).
B. Moving loop in static B field (Motional EMF)

- It is kind of emf found in electrical machines such as motors and generators.
- Given below is an example of a dc machine, where voltage is generated as the coil rotates within the magnetic field.
B. Moving loop in static B field (Motional EMF)

- Another example of motional emf is illustrated below, where a conducting bar is moving between a pair of rails.
B. Moving loop in static B field (Motional EMF)

* Recall that the force on a charge moving with uniform velocity \( u \) in a magnetic field \( B \) is: \( F_m = Qu \times B \)

* We define the motional electric field \( E_m \) as

\[
E_m = \frac{F_m}{Q} = u \times B
\]

\[
V_{\text{emf}} = \oint E_m \cdot dl = \oint (u \times B) \cdot dl
\]

By applying Stoke's theorem,

\[
\int_S (\nabla \times E_m) \cdot dS = \oint_S \nabla \times (u \times B) \cdot dS
\]

or \( \nabla \times E_m = \nabla \times (u \times B) \)
B. Moving loop in static B field (Motional EMF)

\[ V_{\text{emf}} = \oint E_m \cdot dl = \oint (u \times B) \cdot dl \]

Notes:

- The integral is zero along the portion of the loop where \( u = 0 \). (e.g. \( dl \) is taken along the rod in the shown figure.)
- The direction of the induced current is the same that of \( E_m \) or \( u \times B \). The limits of integration are selected in the direction opposite to the direction of \( u \times B \) to satisfy Lenz’s law. (e.g. induced current flows in the rod along \( ay \), the integration over \( L \) is along \(-ay\).)
Both Transformer emf and motional emf are present.

\[ V_{\text{emf}} = \oint_{L} E \cdot dl = -\int_{S} \frac{\partial B}{\partial t} \cdot dS + \oint_{L} (u \times B) \cdot dl \]

or

\[ \nabla \times E = -\frac{\partial B}{\partial t} + \nabla \times (u \times B) \]

Note that eq. \[ V_{\text{emf}} = -\frac{d\Psi}{dt} \] can always be applied in place of the equations in cases A, B, and C.
a conducting bar can slide freely over two conducting rails as shown in the figure. Calculate the induced voltage in the bar.

(a) If the bar is stationed at $y=8 \text{ cm}$ and $B = 4 \cos 10^6 t \ a_z \ \text{mWb/m}^2$

(b) If the bar slides at a velocity $u = 20 \ a_y \ \text{m/s}$ and $B = 4a_z \ \text{mWb/m}^2$

(c) If the bar slid at a velocity $u = 20 \ a_y \ \text{m/s}$ and $B = 4 \cos \left(10^6 t - y\right) \ a_z \ \text{mWb/m}^2$
(a) we have transformer emf

\[
V_{\text{emf}} = -\int \frac{\partial B}{\partial t} \cdot dS = \int \int 4(10^{-3})(10^6) \sin 10^6 t \, dx \, dy
\]

\[
= 4(10^3)(0.08)(0.06) \sin 10^6 t
\]

\[
= 19.2 \sin 10^6 t \, V
\]

(b) This is motional emf

\[
V_{\text{emf}} = \int (u \times B) \cdot dl = \int_{x=l}^{0} (ua_y \times Ba_z) \cdot dx \, a_x
\]

\[
= -UBL = -20(4.10^{-3})(0.06)
\]

\[
= -4.8 \, \text{mV}
\]
(c) Both transformer emf and motional emf are present in this case.

This problem can be solved in two ways:

Method 1:

\[ V_{\text{emf}} = - \int \frac{\partial B}{\partial t} \cdot dS + \int (u \times B) \cdot dl \]

\[ = \int \int 4 \cdot (10^{-3}) (10^6) \sin(10^6 t - y) \ dy \ dx \]

\[ + \int \int \left[ 20a_y \times 4 \cdot 10^{-3} \cos(10^6 t - y) a_z \right] \ dx \ ax \]

\[ = 240 \cos \left( 10^6 t - y \right) \bigg|_{0}^{y} - 80 (10^{-3}) (0.06) \cos(10^6 t - y) \]

\[ = 240 \cos \left( 10^6 t - y \right) - 240 \cos 10^6 t - 4.8(10^{-3}) \cos \left( 10^6 t - y \right) \]

\[ \approx 240 \cos \left( 10^6 t - y \right) - 240 \cos 10^6 t \]
Method 2:

\[ \text{Vemf} = -\frac{\partial \Psi}{\partial t}, \quad \text{where} \quad \Psi = \int B.\,dS \]

\[
= \int_{y=0}^{0.06} \int_{x=0}^{y} 4 \cos(10^6 t - y) \, dx \, dy
\]

\[
= -40(0.06) \sin(10^6 t - y) \bigg|_{y=0}^{y}
\]

\[
= -0.24 \sin(10^6 t - y) + 0.24 \sin 10^6 t \text{ mWb}
\]

But

\[
\frac{dy}{dt} = u \rightarrow y = ut = 20t
\]

Hence,

\[
\Psi = -0.24 \sin(10^6 t - 20t) + 0.24 \sin 10^6 t \text{ mWb}
\]

\[
\text{Vemf} = -\frac{\partial \Psi}{\partial t} = 0.24(10^6 - 20) \cos(10^6 t - 20t) - 0.24(10^6) \cos 10^6 t \text{ mV}
\]

\[
= 240 \cos \left(10^6 t - y\right) - 240 \cos 10^6 t
\]
Example 9.2

The loop shown in the figure is inside a uniform magnetic field $B = 50 \text{ a}_x \text{ mW/m}^2$. If side DC of the loop cuts the flux lines at the frequency of 50 Hz and the loop lies in the yz-plane at the time $t = 0$, find

(a) The induced emf at $t = 1 \text{ ms}$.

(b) The induced current at $t = 3 \text{ ms}$.
The B field is time invariant, the induced emf is motional

\[ V_{\text{emf}} = \int (u \times B).dl, \quad \text{where} \quad dl = d_z a_z, \quad u = \frac{\rho \partial \phi}{\partial t} a_\phi = \rho \omega a_\phi \]

\( \rho = AD = 4 \text{ cm}, \ \omega = 2\pi f = 100\pi \)

Transform B into cylindrical coordinates:

\[ B = B_0 a_x = B_0 \left( \cos \phi a_\rho - \sin \phi a_\phi \right), \quad \text{where} \quad B_0 = 0.05 \]

\[
\begin{vmatrix}
\begin{array}{ccc}
\rho \\ 0 \\ B_0 \cos \phi
\end{array}
\end{vmatrix}
\begin{vmatrix}
\begin{array}{ccc}
a_\rho \\ a_\phi \\ a_z
\end{array}
\end{vmatrix}
\begin{vmatrix}
\begin{array}{c}
\rho \omega \\
0 \\
-B_0 \sin \phi
\end{array}
\end{vmatrix}
\]

\[
\begin{vmatrix}
\begin{array}{ccc}
\rho \omega & 0 & 0 \\
B_0 \cos \phi & -B_0 \sin \phi & 0
\end{array}
\end{vmatrix}
\]

\[
(u \times B).dl = -\rho \omega B_0 \cos \phi \, dz = -0.04(100\pi)(0.05) \cos \phi \, dz
\]

\[ = -0.2\pi \cos \phi \, dz \]

\[ V_{\text{emf}} = \int_{z=0}^{0.03} -0.2\pi \cos \phi \, dz = -6\pi \cos \phi \, \text{mV} \]
To determine $\phi$ recall that

$$\omega = \frac{d\phi}{dt} \rightarrow \phi = \omega t + C_0 \quad (C_0 \text{ is the integration constant})$$

At $t=0$, $\phi = \pi / 2$ because the loop is in the yz-plane at that time,

$$C_0 = \pi / 2. \text{ Hence, } \phi = \omega t + \pi / 2$$

$$V_{emf} = -6\pi \cos(\omega t + \pi / 2) = 6\pi \sin(100\pi t) \text{ mV}$$

At $t=1 \text{ ms}$, $V_{emf} = 6\pi \sin(0.1\pi) = 5.825 \text{ mV}$

(b) The current induced is

$$i = \frac{V_{emf}}{R} = 60\pi \sin(100\pi t) \text{ mA}$$

At $t=3\text{ ms}$, $i = 60\pi \sin(0.3\pi) \text{ mA} = 0.1525 \text{ A}$

Note that for sides AD and BC $(u \times B).dl = 0$ since $a_z.a_\rho = 0$
9.4 Displacement current

Here we will consider Maxwell's curl equation for magnetic fields (Ampere's Law) for time-varying conditions.

For static EM fields, recall that

\[ \nabla \times H = J \quad \ldots \quad (1) \]

But the divergence of the curl is zero:

\[ \nabla \cdot (\nabla \times H) = 0 = \nabla \cdot J \quad \ldots \quad (2) \]

However, the equation of continuity requires that

\[ \nabla \cdot J = -\frac{\partial \rho_v}{\partial t} \quad \ldots \quad (3) \]

Thus, equations (2) and (3) are incompatible for time varying conditions!!
Here we must modify equation (1) to consider time-varying situation:

To do this, add a term \( J_d \) to eq. (1):

\[
\nabla \times H = J + J_d
\]

... (4)

again, taking the divergence, we have:

\[
\nabla \cdot (\nabla \times H) = 0 = \nabla \cdot J + \nabla \cdot J_d
\]

... (5)

Since \( \nabla \cdot J = -\frac{\partial \rho_v}{\partial t} \), then, \( \nabla \cdot J_d = \frac{\partial \rho_v}{\partial t} \)

Since \( \nabla \cdot D = \rho_v \)

\[
\rightarrow \nabla \cdot J_d = \frac{\partial}{\partial t} (\nabla \cdot D) = \nabla \cdot \frac{\partial D}{\partial t} \rightarrow J_d = \frac{\partial D}{\partial t}
\]

... (6)

Substituting eq (6) into eq (4) results in,

\[
\nabla \times H = J + \frac{\partial D}{\partial t}
\]

... (7)
Displacement current

\[ \nabla \times H = J + \frac{\partial D}{\partial t} \]

* This is Maxwell's equation (based on Ampere's circuit law) for a time varying field.

The term \( J_d = \frac{\partial D}{\partial t} \) is known as displacement current density.

This is the third type of current density we have met:

- **Conduction current density:** \( J = \sigma E \)
  (motion of charge in a conductor)

- **Convection Current Density:** \( J = \rho_v u \)
  (doesn’t involve conductors, current flows through an insulating medium, such as liquid, or vacuum).

- **Displacement Current Density:** \( J_d = \frac{\partial D}{\partial t} \)
  (is a result of time-varying electric field).
The insertion of $J_d$ into Ampere’s equation was one of the major contributions of Maxwell.

Without the term $J_d$, the propagation of electromagnetic waves (e.g., radio or TV) would be impossible.

Displacement current is the mechanism which allows electromagnetic waves to propagate in a non-conducting medium.

The displacement current is defined as:

$$I_d = \int J_d \cdot dS = \int \frac{\partial D}{\partial t} \cdot dS$$
Displacement current

A typical example of displacement current is the current through a capacitor when alternating voltage source is applied to its plates.

- The total current density is $J + J_d$.
- Take Amperian path as shown. Consider 2 surfaces bounded by path $L$.

**If surface $S_1$ is chosen:** $J_d = 0$

$$\oint_{L} \mathbf{H}.d\mathbf{l} = \int_{S_1} \mathbf{J}.d\mathbf{S} = I_{enc} = I$$

**If surface $S_2$ is chosen:** $J = 0$

$$\oint_{L} \mathbf{H}.d\mathbf{l} = \int_{S_2} J_d.d\mathbf{S} = \frac{d}{dt} \int_{S_2} \mathbf{D}.d\mathbf{S} = \frac{dQ}{dt} = I$$

- So we obtain the same current for either surface.
- A time-varying electric field induces magnetic field inside the capacitor.
Conduction to Displacement Current Ratio

The conduction current density is given by $J_c = \sigma E$

The displacement current density is given by $J_d = \varepsilon \frac{\partial E}{\partial t}$

Assume that the electric field is a sinusoidal function of time:

$E = E_0 \cos \omega t$

Then, $J_c = \sigma E_0 \cos \omega t$, $J_d = -\omega \varepsilon E_0 \sin \omega t$

We have $|J_c|_{\text{max}} = \sigma E_0$, $|J_d|_{\text{max}} = \omega \varepsilon E_0$

Therefore $\frac{|J_c|_{\text{max}}}{|J_d|_{\text{max}}} = \frac{\sigma}{\omega \varepsilon}$

$\sigma >> \omega \varepsilon$ Good Conductor ($I_d$ negligible)

$\sigma << \omega \varepsilon$ Good Insulator ($I_c$ negligible)

**Note:** In free space (or other perfect dielectric), the conduction current is zero and only displacement current can exist.
Example 9.4

A parallel plate capacitor with plate area of 5 cm\(^2\) and plate separation of 3 mm has a voltage 50 \(\sin 10^3 t\) V applied to its plates. Calculate the displacement current assuming \(\varepsilon = 2\varepsilon_0\).

\[
I_d = J_d \cdot S, \quad J_d = \frac{\partial D}{\partial t}
\]

but \(D = \varepsilon E = \varepsilon \frac{V}{d}\) \(\rightarrow\) \(J_d = \frac{\partial D}{\partial t} = \frac{\varepsilon}{d} \frac{dV}{dt}\)

\[
\rightarrow I_d = \frac{\varepsilon S}{d} \frac{dV}{dt} = C \frac{dV}{dt} \quad \text{(same as conduction current } I_c).\)

\[
\rightarrow I_d = 2 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{5 \times 10^{-4}}{3 \times 10^{-3}} \cdot 10^{-3} \times 50 \cos(10^3 t)
\]

\[
\rightarrow I_d = 147.4 \cos(10^3 t) \text{ nA}
\]
### 9.5 Maxwell’s Equations in Final Forms

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<td>Ampere's Law modified by continuity eqn</td>
</tr>
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The concepts of linearity, isotropy, and homogeneity of a material medium still apply for time varying fields.

In a linear, homogeneous, and isotropic medium characterized by \( \sigma, \varepsilon, \) and \( \mu \), the following relations hold for time varying fields:

\[
\begin{align*}
D &= \varepsilon E = \varepsilon_0 E + P \\
B &= \mu H = \mu_0 (H + M) \\
J &= \sigma E + \rho_V u
\end{align*}
\]

Consequently, the boundary conditions remain valid for time varying fields.

\[
\begin{align*}
E_{1t} - E_{2t} &= 0, \\
D_{1n} - D_{2n} &= \rho_S \\
H_{1t} - H_{2t} &= K, \\
B_{1n} - B_{2n} &= 0
\end{align*}
\]
Electromagnetic flow diagrams

Figure 9.11 Electromagnetic flow diagrams showing the relationship between the potentials and vector fields: (a) electrostatic system, (b) magnetostatic system, (c) electromagnetic system.