Chapter 17

Least Square Regression
Part 5 - CURVE FITTING

Describes techniques to fit curves (curve fitting) to discrete data to obtain intermediate estimates.

There are two general approaches for curve fitting:

• **Least Squares regression:**
  Data exhibit a significant degree of scatter. The strategy is to derive a single curve that represents the general trend of the data.

• **Interpolation:**
  Data is very precise. The strategy is to pass a curve or a series of curves through each of the points.
Introduction

In engineering, two types of applications are encountered:

– **Trend analysis.** Predicting values of dependent variable, may include extrapolation beyond data points or interpolation between data points.

– **Hypothesis testing.** Comparing existing mathematical model with measured data.
Mathematical Background

• **Arithmetic mean.** The sum of the individual data points \((y_i)\) divided by the number of points \((n)\).

\[
\bar{y} = \frac{\sum y_i}{n}, \quad i = 1, \ldots, n
\]

• **Standard deviation.** The most common measure of a spread for a sample.

\[
S_y = \sqrt{\frac{S_t}{n-1}}, \quad S_t = \sum (y_i - \bar{y})^2
\]
**Mathematical Background (cont’d)**

- **Variance**. Representation of spread by the square of the standard deviation.
  
  \[ S_y^2 = \frac{\sum (y_i - \bar{y})^2}{n-1} \]

  \[ S_y^2 = \frac{\sum y_i^2 - (\sum y_i)^2}{n-1} \]

- **Coefficient of variation**. Has the utility to quantify the spread of data.
  
  \[ c.v. = \frac{S_y}{\bar{y}} \times 100\% \]
Chapter 17
Least Squares Regression

Linear Regression
Fitting a straight line to a set of paired observations: \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\).

\[ y = a_0 + a_1 x + e \]

- \(a_1\) - slope
- \(a_0\) - intercept
- \(e\) - error, or residual, between the model and the observations
Linear Regression: Residual

\[ y_i - a_0 - a_1 x_i \]

\[ a_0 + a_1 x_i \]
Linear Regression: Question

How to find $a_0$ and $a_1$ so that the error would be minimum?
Linear Regression: Criteria for a “Best” Fit

\[
\min \sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)
\]

\[e_1 = -e_2\]
Linear Regression: Criteria for a “Best” Fit

\[ \min \sum_{i=1}^{n} | e_i | = \sum_{i=1}^{n} | y_i - a_0 - a_1 x_i | \]
Linear Regression: Criteria for a “Best” Fit

\[
\min_{i=1}^{n} \max \left| e_i \right| = \left| y_i - a_0 - a_1 x_i \right|
\]
Linear Regression: Least Squares Fit

\[ S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i, \text{measured} - y_i, \text{model})^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2 \]

\[
\min S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2
\]

Yields a unique line for a given set of data.
Linear Regression: Least Squares Fit

\[
\min S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2
\]

The coefficients \(a_0\) and \(a_1\) that minimize \(S_r\) must satisfy the following conditions:

\[
\begin{align*}
\frac{\partial S_r}{\partial a_0} &= 0 \\
\frac{\partial S_r}{\partial a_1} &= 0
\end{align*}
\]
Linear Regression: Determination of $a_o$ and $a_1$

\[
\frac{\partial S_r}{\partial a_o} = -2 \sum (y_i - a_o - a_1 x_i) = 0
\]

\[
\frac{\partial S_r}{\partial a_1} = -2 \sum [(y_i - a_o - a_1 x_i) x_i] = 0
\]

\[
0 = \sum y_i - \sum a_o - \sum a_1 x_i
\]

\[
0 = \sum y_i x_i - \sum a_o x_i - \sum a_1 x_i^2
\]

\[
\sum a_o = na_o
\]

\[
na_o + (\sum x_i) a_1 = \sum y_i
\]

\[
\sum y_i x_i = \sum a_o x_i + \sum a_1 x_i^2
\]

\[
\text{2 equations with 2 unknowns, can be solved simultaneously}
\]
Linear Regression:
Determination of $a_0$ and $a_1$

\[
a_1 = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}
\]

\[
a_0 = \bar{y} - a_1 \bar{x}
\]
A diagram illustrating a linear regression model. The regression line is represented by the equation $y = a_0 + a_1 x_i$. The measurement $y_i$ is the observed value at $x_i$, and the residual is $y_i - a_0 - a_1 x_i$.
Data spread around Mean

Data spread around best-fit line
Examples of linear regression with (a) small and (b) large residual errors
Error Quantification of Linear Regression

- Total sum of the squares around the mean for the dependent variable, $y$, is $S_t$
  \[ S_t = \sum (y_i - \bar{y})^2 \]

- Sum of the squares of residuals around the regression line is $S_r$
  \[ S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_o - a_1 x_i)^2 \]
Error Quantification of Linear Regression

• $S_t - S_r$ quantifies the improvement or error reduction due to describing data in terms of a straight line rather than as an average value.

$$r^2 = \frac{S_t - S_r}{S_t}$$

$r^2$: coefficient of determination

$r$: correlation coefficient
Error Quantification of Linear Regression

For a perfect fit:

• $S_r = 0$ and $r = r^2 = 1$, signifying that the line explains 100 percent of the variability of the data.

• For $r = r^2 = 0$, $S_r = S_t$, the fit represents no improvement.
Least Squares Fit of a Straight Line: Example

Fit a straight line to the x and y values in the following Table:

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$y_i$</th>
<th>$x_i y_i$</th>
<th>$x_i^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2.5</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>3.5</td>
<td>17.5</td>
<td>25</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>36</td>
<td>36</td>
</tr>
<tr>
<td>7</td>
<td>5.5</td>
<td>38.5</td>
<td>49</td>
</tr>
</tbody>
</table>

$\sum x_i = 28 \quad \sum y_i = 24.0$

$\sum x_i^2 = 140 \quad \sum x_i y_i = 119.5$

$\bar{x} = \frac{28}{7} = 4$

$\bar{y} = \frac{24}{7} = 3.428571$
Least Squares Fit of a Straight Line: Example (cont’d)

\[ a_1 = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2} \]

\[ = \frac{7 \times 119.5 - 28 \times 24}{7 \times 140 - 28^2} = 0.8392857 \]

\[ a_0 = \bar{y} - a_1 \bar{x} \]

\[ = 3.428571 - 0.8392857 \times 4 = 0.07142857 \]

\[ Y = 0.07142857 + 0.8392857 \times x \]
## Least Squares Fit of a Straight Line: Example (Error Analysis)

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$y_i$</th>
<th>$(y_i - \bar{y})^2$</th>
<th>$e_i^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>8.5765</td>
<td>0.1687</td>
</tr>
<tr>
<td>2</td>
<td>2.5</td>
<td>0.8622</td>
<td>0.5625</td>
</tr>
<tr>
<td>3</td>
<td>2.0</td>
<td>2.0408</td>
<td>0.3473</td>
</tr>
<tr>
<td>4</td>
<td>4.0</td>
<td>0.3265</td>
<td>0.3265</td>
</tr>
<tr>
<td>5</td>
<td>3.5</td>
<td>0.0051</td>
<td>0.5896</td>
</tr>
<tr>
<td>6</td>
<td>6.0</td>
<td>6.6122</td>
<td>0.7972</td>
</tr>
<tr>
<td>7</td>
<td>5.5</td>
<td>4.2908</td>
<td>0.1993</td>
</tr>
</tbody>
</table>

$S_t = \sum (y_i - \bar{y})^2 = 22.7143$

$S_r = \sum e_i^2 = 2.9911$

$r^2 = \frac{S_t - S_r}{S_t} = 0.868$

$r = \sqrt{r^2} = \sqrt{0.868} = 0.932$

$Y = 0.07142857 + 0.8392857 x$

$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_o - a_l x_i)^2$
Least Squares Fit of a Straight Line: Example (Error Analysis)

• The standard deviation (quantifies the spread around the mean):
\[ s_y = \sqrt{\frac{S_t}{n-1}} = \sqrt{\frac{22.7143}{7-1}} = 1.9457 \]

• The standard error of estimate (quantifies the spread around the regression line)
\[ s_{y/x} = \sqrt{\frac{S_r}{n-2}} = \sqrt{\frac{2.9911}{7-2}} = 0.7735 \]

Because \( s_{y/x} < s_y \), the linear regression model has good fitness
Algorithm for linear regression

SUB Regress(x, y, n, a1, a0, syx, r2)

\[\text{sumx} = 0: \text{sumxy} = 0: \text{st} = 0\]
\[\text{sumy} = 0: \text{sumx2} = 0: \text{sr} = 0\]
DO \(i = 1, n\)
\[\text{sumx} = \text{sumx} + x_i\]
\[\text{sumy} = \text{sumy} + y_i\]
\[\text{sumxy} = \text{sumxy} + x_i y_i\]
\[\text{sumx2} = \text{sumx2} + x_i^2\]
END DO
\[\text{xm} = \text{sumx}/n\]
\[\text{ym} = \text{sumy}/n\]
\[a1 = (n \times \text{sumxy} - \text{sumx} \times \text{sumy})/(n \times \text{sumx2} - \text{sumx}^2)\]
\[a0 = \text{ym} - a1 \times \text{xm}\]
DO \(i = 1, n\)
\[\text{st} = \text{st} + (y_i - \text{ym})^2\]
\[\text{sr} = \text{sr} + (y_i - a1 \times x_i - a0)^2\]
END DO
\[\text{syx} = (\text{sr}/(n - 2))^{0.5}\]
\[r2 = (\text{st} - \text{sr})/\text{st}\]

END Regress
Linearization of Nonlinear Relationships

• The relationship between the dependent and independent variables is linear.

• However, few types of nonlinear functions can be transformed into linear regression problems.
  ➢ The exponential equation.
  ➢ The power equation.
  ➢ The saturation-growth-rate equation.
The exponential equation

The power equation

Saturation-growth-rate equation

(a) Linearization
(b) Linearization
(c) Linearization

\[ y = a_1 e^{b_1 x} \]

\[ y = a_2 x^{b_2} \]

\[ y = a_3 \frac{x}{b_3 + x} \]

\[ \ln y = \ln a_1 + b_1 x \]

\[ \log y = \log a_2 + b_2 \log x \]

\[ \frac{1}{y} = \frac{1}{a_3} + \frac{b_3}{a_3} \cdot \frac{1}{x} \]

Intercept = \ln a_1

Slope = b_1

Intercept = \log a_2

Slope = b_2

Intercept = \log 1/a_3

Slope = b_3/a_3
Linearization of Nonlinear Relationships

1. The exponential equation.

\[ y = a_1 e^{b_1 x} \]

\[ \ln y = \ln a_1 + b_1 x \]
Linearization of Nonlinear Relationships

2. The power equation

\[ y = a_2 x^{b_2} \]

\[ \log y = \log a_2 + b_2 \log x \]
Linearization of Nonlinear Relationships

3. The saturation-growth-rate equation

\[ y = a_3 \frac{x}{b_3 + x} \]

\[ \frac{1}{y} = \frac{1}{a_3} + \frac{b_3}{a_3} \left( \frac{1}{x} \right) \]
Example

Fit the following Equation:

\[ y = a_2 x^{b_2} \]

to the data in the following table:

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( y_i )</th>
<th>( X = \log x_i )</th>
<th>( Y = \log y_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0</td>
<td>-0.301</td>
</tr>
<tr>
<td>2</td>
<td>1.7</td>
<td>0.301</td>
<td>0.226</td>
</tr>
<tr>
<td>3</td>
<td>3.4</td>
<td>0.477</td>
<td>0.534</td>
</tr>
<tr>
<td>4</td>
<td>5.7</td>
<td>0.602</td>
<td>0.753</td>
</tr>
<tr>
<td>5</td>
<td>8.4</td>
<td>0.699</td>
<td>0.922</td>
</tr>
<tr>
<td>15</td>
<td>19.7</td>
<td>2.079</td>
<td>2.141</td>
</tr>
</tbody>
</table>

\[ \log y = \log(a_2 x^{b_2}) \]

\[ \log y = \log a_2 + b_2 \log x \]

let \( Y = \log y \), \( X = \log x \),

\[ a_0 = \log a_2, \ a_1 = b_2 \]

\[ Y = a_0 + a_1 X \]
### Example

<table>
<thead>
<tr>
<th>Xi</th>
<th>Yi</th>
<th>(X^*_{i}=\log(X))</th>
<th>(Y^*_{i}=\log(Y))</th>
<th>(X^<em>Y^</em>)</th>
<th>(X^*^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.0000</td>
<td>-0.3010</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>1.7</td>
<td>0.3010</td>
<td>0.2304</td>
<td>0.0694</td>
<td>0.0906</td>
</tr>
<tr>
<td>3</td>
<td>3.4</td>
<td>0.4771</td>
<td>0.5315</td>
<td>0.2536</td>
<td>0.2276</td>
</tr>
<tr>
<td>4</td>
<td>5.7</td>
<td>0.6021</td>
<td>0.7559</td>
<td>0.4551</td>
<td>0.3625</td>
</tr>
<tr>
<td>5</td>
<td>8.4</td>
<td>0.6990</td>
<td>0.9243</td>
<td>0.6460</td>
<td>0.4886</td>
</tr>
<tr>
<td><strong>Sum</strong></td>
<td><strong>15</strong></td>
<td><strong>19.700</strong></td>
<td><strong>2.079</strong></td>
<td><strong>2.141</strong></td>
<td><strong>1.424</strong></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
  a_1 &= \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2} = \frac{5 \times 1.424 - 2.079 \times 2.141}{5 \times 1.169 - 2.079^2} = 1.75 \\
  a_0 &= \bar{y} - a_1 \bar{x} = 0.4282 - 1.75 \times 0.41584 = -0.334
\end{align*}
\]
Linearization of Nonlinear Functions: Example

\[ \log y = -0.334 + 1.75 \log x \]

\[ y = 0.46x^{1.75} \]
Polynomial Regression

• Some engineering data is poorly represented by a straight line.
• For these cases a curve is better suited to fit the data.
• The least squares method can readily be extended to fit the data to higher order polynomials.
A parabola is preferable
Polynomial Regression (cont’d)

- A 2\textsuperscript{nd} order polynomial (quadratic) is defined by:

\[ y = a_o + a_1 x + a_2 x^2 + e \]

- The residuals between the model and the data:

\[ e_i = y_i - a_o - a_1 x_i - a_2 x_i^2 \]

- The sum of squares of the residual:

\[ S_r = \sum e_i^2 = \sum \left( y_i - a_o - a_1 x_i - a_2 x_i^2 \right)^2 \]
Polynomial Regression (cont’d)

\[
\frac{\partial S_r}{\partial a_o} = -2 \sum (y_i - a_o - a_1 x_i - a_2 x_i^2) = 0
\]

\[
\frac{\partial S_r}{\partial a_1} = -2 \sum (y_i - a_o - a_1 x_i - a_2 x_i^2) x_i = 0
\]

\[
\frac{\partial S_r}{\partial a_2} = -2 \sum (y_i - a_o - a_1 x_i - a_2 x_i^2) x_i^2 = 0
\]

\[
\sum y_i = n \cdot a_o + a_1 \sum x_i + a_2 \sum x_i^2
\]

\[
\sum x_i y_i = a_o \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3
\]

\[
\sum x_i^2 y_i = a_o \sum x_i^2 + a_1 \sum x_i^3 + a_2 \sum x_i^4
\]

3 linear equations with 3 unknowns \((a_o, a_1, a_2)\), can be solved
Polynomial Regression (cont’d)

- A system of 3x3 equations needs to be solved to determine the coefficients of the polynomial.

\[
\begin{bmatrix}
  n & \sum x_i & \sum x_i^2 \\
  \sum x_i & \sum x_i^2 & \sum x_i^3 \\
  \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \\
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
\end{bmatrix}
= \begin{bmatrix}
\sum y_i \\
\sum x_i y_i \\
\sum x_i^2 y_i \\
\end{bmatrix}
\]

- The standard error & the coefficient of determination

\[
s_{y/x} = \sqrt{\frac{S_r}{n-3}}
\]

\[
r^2 = \frac{S_t - S_r}{S_t}
\]
Polynomial Regression (cont’d)

General:

The mth-order polynomial:

\[ y = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m + e \]

• A system of \((m+1)\times(m+1)\) linear equations must be solved for determining the coefficients of the mth-order polynomial.

• The standard error:

\[ s_{y/x} = \sqrt{\frac{S_r}{n-(m+1)}} \]

• The coefficient of determination:

\[ r^2 = \frac{S_t - S_r}{S_t} \]
### Polynomial Regression - Example

Fit a second order polynomial to data:

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$y_i$</th>
<th>$x_i^2$</th>
<th>$x_i^3$</th>
<th>$x_i^4$</th>
<th>$x_i y_i$</th>
<th>$x_i^2 y_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>7.7</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>7.7</td>
<td>7.7</td>
</tr>
<tr>
<td>2</td>
<td>13.6</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>27.2</td>
<td>54.4</td>
</tr>
<tr>
<td>3</td>
<td>27.2</td>
<td>9</td>
<td>27</td>
<td>81</td>
<td>81.6</td>
<td>244.8</td>
</tr>
<tr>
<td>4</td>
<td>40.9</td>
<td>16</td>
<td>64</td>
<td>256</td>
<td>163.6</td>
<td>654.4</td>
</tr>
<tr>
<td>5</td>
<td>61.1</td>
<td>25</td>
<td>125</td>
<td>625</td>
<td>305.5</td>
<td>1527.5</td>
</tr>
<tr>
<td>15</td>
<td>152.6</td>
<td>55</td>
<td>225</td>
<td>979</td>
<td>585.6</td>
<td>2489</td>
</tr>
</tbody>
</table>

$$
\begin{align*}
\sum x_i &= 15 \\
\sum y_i &= 152.6 \\
\sum x_i^2 &= 55 \\
\sum x_i^3 &= 225 \\
\sum x_i^4 &= 979 \\
\sum x_i y_i &= 585.6 \\
\sum x_i^2 y_i &= 2489.8
\end{align*}
$$

$$
\bar{x} = \frac{15}{6} = 2.5, \quad \bar{y} = \frac{152.6}{6} = 25.433
$$
Polynomial Regression - Example (cont’d)

- The system of simultaneous linear equations:

\[
\begin{bmatrix}
6 & 15 & 55 \\
15 & 55 & 225 \\
55 & 225 & 979
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2
\end{bmatrix}
= 
\begin{bmatrix}
152.6 \\
585.6 \\
2488.8
\end{bmatrix}
= 
\begin{bmatrix}
\sum x_i \\
\sum x_i^2 \\
\sum x_i^3
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2
\end{bmatrix}
+ 
\begin{bmatrix}
\sum x_i y_i \\
\sum x_i^2 y_i
\end{bmatrix}
\]

\[
a_0 = 2.47857, 
 a_1 = 2.35929, 
 a_2 = 1.86071
\]

\[
y = 2.47857 + 2.35929x + 1.86071x^2
\]
### Polynomial Regression - Example (cont’d)

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$y_i$</th>
<th>$Y_{model}$</th>
<th>$e_i^2$</th>
<th>$(y_i-y^')^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.1</td>
<td>2.4786</td>
<td>0.14332</td>
<td>544.42889</td>
</tr>
<tr>
<td>1</td>
<td>7.7</td>
<td>6.6986</td>
<td>1.00286</td>
<td>314.45929</td>
</tr>
<tr>
<td>2</td>
<td>13.6</td>
<td>14.64</td>
<td>1.08158</td>
<td>140.01989</td>
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<tr>
<td>3</td>
<td>27.2</td>
<td>26.303</td>
<td>0.80491</td>
<td>3.12229</td>
</tr>
<tr>
<td>4</td>
<td>40.9</td>
<td>41.687</td>
<td>0.61951</td>
<td>239.22809</td>
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<tr>
<td>5</td>
<td>61.1</td>
<td>60.793</td>
<td>0.09439</td>
<td>1272.13489</td>
</tr>
<tr>
<td>15</td>
<td>152.6</td>
<td>3.74657</td>
<td>2513.39333</td>
<td></td>
</tr>
</tbody>
</table>

- The standard error of estimate:

$$s_{y/x} = \sqrt{\frac{3.74657}{6-3}} = 1.12$$

- The coefficient of determination:

$$r^2 = \frac{2513.39 - 3.74657}{2513.39} = 0.99851, \quad r = \sqrt{r^2} = 0.99925$$

![Least-squares parabola](image)
**Nonlinear Regression**

- Consider the previous exponential regression:

  \[ y = f(x_i) = a_o (1 - e^{-a_1 x}) \]

- The sum of the squares of the residuals:

  \[ S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(y_i - a_o (1 - e^{-a_1 x_i}) \right)^2 = \sum_{i=1}^{n} \left(y_i - f(x_i) \right)^2 \]

- The criterion for least squares regression is:

  \[ \frac{\partial S_r}{\partial a_o} = 0 \quad \& \quad \frac{\partial S_r}{\partial a_1} = 0 \]
Nonlinear Regression

\[ y = f(x_i) = a_o \left( 1 - e^{-a_1 x} \right) \]

\[ S_r = \sum_{i=1}^{n} \left( y_i - a_o \left( 1 - e^{-a_1 x_i} \right) \right)^2 = \sum_{i=1}^{n} \left( y_i - f(x_i) \right)^2 \]

\[ \frac{\partial S_r}{\partial a_o} = 0 \quad \& \quad \frac{\partial S_r}{\partial a_1} = 0 \]

\[ \frac{\partial S_r}{\partial a_o} = -2 \sum_{i=1}^{n} \left( y_i - f(x_i) \right) \left( \frac{\partial f(x_i)}{\partial a_o} \right) = 0 \]

\[ \frac{\partial S_r}{\partial a_1} = -2 \sum_{i=1}^{n} \left( y_i - f(x_i) \right) \left( \frac{\partial f(x_i)}{\partial a_1} \right) = 0 \]
Nonlinear Regression

\[ \sum_{i=1}^{n} (y_i - f(x_i)) \left( \frac{\partial f(x_i)}{\partial a_o} \right) = 0 \]

\[ \sum_{i=1}^{n} (y_i - f(x_i)) \left( \frac{\partial f(x_i)}{\partial a_1} \right) = 0 \]

- The partial derivatives are expressed at every data point (i) in terms of \( a_o \) and \( a_1 \).
- Thus, the above leads to 2 equations in 2 unknowns which can be solved iteratively for \( a_o \) and \( a_1 \).