Chapter 5: The Normal Distribution (Cont’d)

Section 5.3: Approximating Distributions with the Normal Distribution

Problem (01):
There is a probability of 0.6 that an oyster produces a pearl with a diameter of at least 4 mm, which is consequently of a commercial size.

(a) How many oysters does an oyster farmer need to farm in order to be 99% confident of having at least 1000 pearls of commercial value?

(b) Assume that pearl has an expected diameter of 5 mm and a variance of 8.33. If a farmer collected 1050 pearls, compute the interval over which a farmer can be 99.7% confident of having an average pearl diameter lying within this interval?

(Example 30 in textbook)

Solution:

(a) 99% confident of having at least 1000 pearls:

\[
P = 0.6
\]

\[
n = ?
\]

\[
\mu = nP = 0.6n
\]

\[
\sigma^2 = nP(1 - P) = 0.24n
\]

\[
P(X \geq 1000) \approx P(Y \geq 999.5) \geq 0.9
\]

\[
1 - \Phi \left( \frac{999.5 - 0.6n}{\sqrt{0.24n}} \right) \geq 0.9
\]

\[
\Phi \left( \frac{999.5 - 0.6n}{\sqrt{0.24n}} \right) \leq 0.1
\]

\[
\frac{999.5 - 0.6n}{\sqrt{0.24n}} \leq -2.33
\]

\[
n \leq 1746.8
\]

\[
n = 1750
\]
(b) Compute the interval over which a farmer can be 99.7% confident:

\[
\begin{align*}
\mu &= 5 \\
\sigma^2 &= 8.33 \\
n &= 1050
\end{align*}
\]

For an average pearl diameter:

\[
\begin{align*}
\mu_{\text{av}} &= 5.0 \\
\sigma^2_{\text{av}} &= \frac{\sigma^2}{n} = \frac{8.33}{1050} = 0.00793 \\
\text{So that:}

[ \mu - c \times \sigma, \mu + c \times \sigma ] &= [ 5 - 3 \times \sqrt{0.00793}, 5 + 3 \times \sqrt{0.00793} ] \\
&= [ 4.73, 5.26 ]
\end{align*}
\]

**Problem (02):**

Calculate the following probabilities both exactly and by using a normal approximation:

(a) \( P(X \geq 8) \) where \( X \sim B(10, 0.7) \)

(b) \( P(2 \leq X \leq 7) \) where \( X \sim B(15, 0.3) \)

(c) \( P(X \leq 4) \) where \( X \sim B(9, 0.4) \)

(d) \( P(8 \leq X \leq 11) \) where \( X \sim B(14, 0.6) \)

(Problem 5.3.1 in textbook)

**Solution:**

(a) \( P(X \geq 8) \) where \( X \sim B(10, 0.7) \):

Exact solution:

\[
\begin{align*}
P \left( Z \geq \frac{8 - 10}{\sqrt{0.7}} \right) &= 1.0 - P \left( Z \leq \frac{8 - 10}{\sqrt{0.7}} \right) \\
&= 1.0 - P(Z \leq -2.3905) \\
&= 1.0 - \Phi(-2.3905) \\
&= 1.0 - 0.6177 \\
&= 0.3823
\end{align*}
\]

Approximate solution:

\[
\begin{align*}
E(X) &= \mu = np = 10 \times 0.7 = 7.0 \\
\text{Var}(X) &= \sigma^2 = np(1 - P) = 10 \times 0.7 \times (1 - 0.7) = 2.1
\end{align*}
\]
\[
\begin{align*}
P \left( Z \geq \frac{8 - 0.5 - 7}{\sqrt{2.1}} \right) &= 1.0 - P \left( Z \leq \frac{8 - 0.5 - 7}{\sqrt{2.1}} \right) \\
&= 1.0 - P(Z \leq 0.3450) \\
&= 1.0 - \Phi(0.3450) \\
&= 1.0 - 0.9635 \\
&= 0.3650
\end{align*}
\]

(b) \( P(2 \leq X \leq 7) \) where \( X \sim B(15, 0.3) \):

Exact solution:
\[
P \left( \frac{2 - 15}{\sqrt{0.3}} \leq Z \leq \frac{7 - 15}{\sqrt{0.3}} \right) = P \left( Z \leq \frac{7 - 15}{\sqrt{0.3}} \right) - P \left( Z \leq \frac{2 - 15}{\sqrt{0.3}} \right)
\]
\[
= P(Z \leq -14.6059) - P(Z \leq -23.7346) \\
= \Phi(-14.6059) - \Phi(-23.7346) \\
= 0.9147
\]

Approximate solution:
\[
E(X) = \mu = np = 15 \times 0.3 = 4.5 \\
\text{Var}(X) = \sigma^2 = np(1 - p) = 15 \times 0.3 \times (1 - 0.3) = 3.15 \\
P \left( \frac{2 - 0.5 - 4.5}{\sqrt{3.15}} \leq Z \leq \frac{7 + 0.5 - 4.5}{\sqrt{3.15}} \right)
\]
\[
= P \left( Z \leq \frac{7 + 0.5 - 4.5}{\sqrt{3.15}} \right) - P \left( Z \leq \frac{2 - 0.5 - 4.5}{\sqrt{3.15}} \right)
\]
\[
= P(Z \leq 1.7321) - P(Z \leq -1.1547) \\
= \Phi(1.7321) - \Phi(-1.1547) \\
= 0.9090
\]

(c) \( P(X \leq 4) \) where \( X \sim B(9, 0.4) \):

Exact solution:
\[
P \left( Z \leq \frac{4 - 9}{\sqrt{0.4}} \right) = P(Z \leq -7.9057)
\]
\[
= \Phi(-7.9057) \\
= 0.7334
\]

Approximate solution:
\[
E(X) = \mu = np = 9 \times 0.4 = 3.6
\]
\[ \text{Var}(X) = \sigma^2 = np(1-p) = 9 \times 0.4 \times (1-0.4) = 2.16 \]

\[ P \left( Z \leq \frac{4 + 0.5 - 3.6}{\sqrt{2.16}} \right) = P(Z \leq 0.3450) = \Phi(0.3450) = 0.6364 \]

(d) \( P(B \leq X \leq 11) \) where \( X \sim B(14, 0.6) \):

**Exact solution:**

\[ P \left( \frac{8 - 14}{\sqrt{0.6}} \leq Z \leq \frac{11 - 14}{\sqrt{0.6}} \right) = P \left( Z \leq \frac{11 - 14}{\sqrt{0.6}} \right) - P \left( Z \leq \frac{8 - 14}{\sqrt{0.6}} \right) \]

\[ = P(Z \leq -3.8730) - P(Z \leq -7.7460) = \Phi(-3.8730) - \Phi(-7.7460) = 0.6527 \]

**Approximate solution:**

\( E(X) = \mu = np = 14 \times 0.6 = 8.4 \)

\[ \text{Var}(X) = \sigma^2 = np(1-p) = 14 \times 0.6 \times (1-0.6) = 3.36 \]

\[ P \left( \frac{8 - 0.5 - 8.4}{\sqrt{3.36}} \leq Z \leq \frac{11 + 0.5 - 8.4}{\sqrt{3.36}} \right) \]

\[ = P \left( Z \leq \frac{11 + 0.5 - 8.4}{\sqrt{3.36}} \right) - P \left( Z \leq \frac{8 - 0.5 - 8.4}{\sqrt{3.36}} \right) \]

\[ = P(Z \leq 1.7898) - P(Z \leq 0.0577) = \Phi(1.7898) - \Phi(0.0577) = 0.6429 \]

**Problem (03):**

Suppose that a fair die is rolled 1000 times.

(a) Estimate the probability that the number of 6s is between 150 and 180.

(b) What is the smallest value of \( n \) for which there is a probability of at least 99% of obtaining at least 50 6s in \( n \) rolls of a fair die?

(Problem 5.3.4 in textbook)

**Solution:**

(a) The probability that the number of 6s is between 150 and 180:

\[ P = 1/6 \]
E(X) = μ = np = 1000 × 1/6 = 166.67
Var(X) = σ^2 = np(1−p) = 1000 × 1/6 × (1−1/6) = 138.89

\[ P \left( \frac{150 - 0.5 - 166.67}{\sqrt{138.89}} \leq Z \leq \frac{180 + 0.5 - 166.67}{\sqrt{138.89}} \right) \]
\[ = P \left( Z \leq \frac{180 + 0.5 - 166.67}{\sqrt{138.89}} \right) - P \left( Z \leq \frac{150 - 0.5 - 166.67}{\sqrt{138.89}} \right) \]
\[ = P(Z \leq -1.4616) - P(Z \leq 1.0922) \]
\[ = \Phi(-1.4616) - \Phi(1.0922) \]
\[ = 0.8072 \]

(b) n for at least 99% of obtaining at least 50 6s in n rolls:

= P = 1/6

E(X) = μ = np = n × 1/6 = n/6
Var(X) = σ^2 = np(1−p) = n × 1/6 × (1−1/6) = 5n/36

\[ P \left( Z \geq \frac{50 - 0.5 - n/6}{\sqrt{5n/36}} \right) \geq 0.99 \]
\[ 1.0 - P \left( Z \leq \frac{50 - 0.5 - n/6}{\sqrt{5n/36}} \right) \geq 0.99 \]
\[ 1.0 - \Phi \left( \frac{50 - 0.5 - n/6}{\sqrt{5n/36}} \right) \geq 0.99 \]
\[ \Phi \left( \frac{50 - 0.5 - n/6}{\sqrt{5n/36}} \right) \leq 0.01 \]
\[ n \geq 402 \]
**Problem (04):**
The number of cracks in a ceramic tile has a Poisson distribution with parameter $\lambda = 2.4$.

(a) How would you approximate the distribution of the total number of cracks in 500 ceramic tiles?

(b) Estimate the probability that there are more than 1250 cracks in 500 ceramic tiles.

(Problem 5.3.5 in textbook)

**Solution:**

(a) Approximate the distribution of cracks in 500 ceramic tiles:

For Poisson distribution:

$X \sim P(2.4)$

$\mu = \sigma^2 = \lambda$

\[
\begin{array}{cccc}
\lambda \text{ (average number of cracks)} & \text{in "x" tiles} \\
2.4 & 1.0 \\
\lambda & 500.0 \\
\end{array}
\]

\[\lambda = \frac{2.4 \times 500.0}{1.0} = 1200.0\]

Therefore:

$X \sim P(1200)$

$\mu = \sigma^2 = \lambda = 1200.0$

For the approximate normal distribution:

$X \sim B(1200, 1200)$

(b) Estimate the probability that there are more than 1250 cracks:

\[
P(X \geq 1250) = 1.0 - P(X \leq 1250)
\]

\[
= 1.0 - P\left(Z \leq \frac{1250 - 1200}{\sqrt{1200}}\right)
\]

\[
= 1.0 - P(Z \leq 1.44)
\]

\[
= 1.0 - \Phi(1.44)
\]

\[
= 1.0 - 0.9251
\]

\[
= 0.0749
\]
Problem (05):
A multiple-choice test consists of a series of questions, each with four possible answers.
(a) If there are 60 questions, estimate the probability that a student who guesses blindly at each question will get at least 30 questions right.
(b) How many questions are needed in order to be 99% confident that a student who guesses blindly at each question scores no more than 35% on the test?

Solution:
(a) The probability that a student who guesses blindly at each question will get at least 30 questions right of 60 questions:
\[ P = \frac{1}{4} = 0.25 \]
\[ E(X) = \mu = np = 60 \times 0.25 = 15.0 \]
\[ \text{Var}(X) = \sigma^2 = np(1-P) = 60 \times 0.25 \times (1 - 0.25) = 11.25 \]
\[ P \left( Z \geq \frac{30 - 0.5 - 15}{\sqrt{11.25}} \right) = 1.0 - P \left( Z \leq \frac{30 - 0.5 - 15}{\sqrt{11.25}} \right) \]
\[ = 1.0 - P(Z \leq 4.3719) \]
\[ = 1.0 - \Phi(4.3719) \]
\[ = 1.0 - 1.0 \]
\[ = 0.0 \]

(b) \( P(B(n, 0.25) \leq 0.35n) \geq 0.99 \):
\[ P = \frac{1}{4} = 0.25 \]
\[ E(X) = \mu = np = n \times \frac{1}{4} = n/4 \]
\[ \text{Var}(X) = \sigma^2 = np(1-P) = n \times \frac{1}{4} \times (1 - \frac{1}{4}) = 3n/16 \]
\[ P \left( Z \leq \frac{0.35n + 0.5 - n/4}{\sqrt{\frac{3n}{16}}} \right) \geq 0.99 \]
\[ \Phi \left( \frac{0.35n + 0.5 - n/4}{\sqrt{\frac{3n}{16}}} \right) \geq 0.99 \]
\[ n \geq 92 \]
Exam Problems (applications on chapter 5 as a whole):

Problem (01):
Adults salmon fish have lengths that are normally distributed with a mean of $\mu = 70$ cm and a standard deviation of $\sigma = 5.4$ cm.

(a) (5 points) What is the probability that an adult salmon fish is longer than 80 cm?

(b) (5 points) What is the value of "c" for which there is 90% probability that an adult salmon has length within $[70 - c, 70 + c]$?

(c) (5 points) If you go fishing with a friend, what is the probability that the first adult salmon you catch is longer than the first adult salmon your friend catches?

(d) (5 points) What is the probability that the average length of the first two adult salmon fish you catch is at least 10 cm longer than the first salmon fish your friend catches?

(e) (5 points) What is the probability that 15 fishes selected at random have an average length less than 72 cm?

(Question 2: (25 points) in Final Exam 2010)

Solution:

(a) The probability that an adult salmon fish is longer than 80 cm:

\[
P(X \geq 80) = 1.0 - P(X \leq 80)
\]

\[
= 1.0 - P\left(Z \leq \frac{80 - 70}{5.4}\right)
\]

\[
= 1.0 - P(Z \leq 1.85)
\]

\[
= 1.0 - \Phi(1.85)
\]

\[
= 1.0 - 0.9678
\]

\[
= 0.0322
\]

(b) The value of "c":

\[
P(70 - c \leq X \leq 70 + c) = 0.9
\]

\[
P\left(\frac{(70 - c) - 70}{5.4} \leq X \leq \frac{(70 + c) - 70}{5.4}\right) = 0.9
\]

\[
P(-0.185c \leq X \leq 0.185c) = 0.9
\]

\[
\Phi(0.185c) - \Phi(-0.185c) = 0.9
\]

... ... eqn. 1

Since $\Phi(x) + \Phi(-x) = 1.0$
\[
\Phi(0.185c) + \Phi(-0.185c) = 1.0 \\
\Phi(-0.185c) = 1.0 - \Phi(0.185c) \ldots \ldots \text{eqn. 2}
\]

Substituting from equation (2) into equation (1) results in:
\[
\Phi(0.185c) - [1.0 - \Phi(0.185c)] = 0.9 \\
\Phi(0.185c) - 1.0 + \Phi(0.185c) = 0.9 \\
2\Phi(0.185c) = 1.9 \\
\Phi(0.185c) = \frac{1.9}{2} \\
\Phi(0.185c) = 0.95 \\
0.185c = 1.645 \\
c = 8.883
\]

(c) The probability that the first adult salmon you catch is longer than the first adult salmon your friend catches:
\[
P(70 - c \leq X \leq 70 + c) = 0.9
\]

\(X_1 = \) a random variable for the first salmon fish I catch
\(X_2 = \) a random variable for the first salmon fish my friend catch

\(Y = X_1 - X_2\)
\(a_1 = 1.0\)
\(a_2 = -1.0\)
\[
\mu_Y = a_1\mu_{X_1} + a_2\mu_{X_2} = 1.0 \times 70 - 1.0 \times 70 = 0.0 \\
\sigma_Y^2 = a_1^2 \times \sigma_{Y_1}^2 + a_2^2 \times \sigma_{Y_2}^2 = (1.0)^2 \times (5.4)^2 + (-1.0)^2 \times (5.4)^2 \\
= 58.32
\]

\[
P(Y \geq 0) = 1.0 - P(Y \leq 0) \\
= 1.0 - P\left( Z \leq \frac{0 - 0}{\sqrt{58.32}} \right) = 1 - P(Z \leq 0) \\
= 1.0 - \Phi(0) \\
= 1.0 - 0.5 \\
= 0.5
\]
(d) The probability that the average length of the first two adult salmon fish you catch is at least 10 cm longer than the first salmon fish your friend catches:

\[ \bar{X} = \frac{X1 - X2}{2} \]

\[ Y = \bar{X} - X3 \]

\[ a_1 = 1.0 \]
\[ a_2 = -1.0 \]

\[ \mu_{\bar{X}} = \frac{70 + 70}{2} = 70 \]

\[ \sigma^2_{\bar{X}} = \frac{\sigma^2}{n} = \frac{(5.4)^2}{2} = 14.58 \]

\[ \mu_Y = a_1 \mu_{X1} + a_2 \mu_{X2} = 1.0 \times 70 - 1.0 \times 70 = 0.0 \]

\[ \sigma^2_Y = a_1^2 \sigma^2_{X1} + a_2^2 \sigma^2_{X2} = (1.0)^2 \times (14.58) + (-1.0)^2 \times (5.4)^2 \]
\[ = 43.74 \]

\[ P(Y \geq 10) = 1.0 - P(Y \leq 10) \]
\[ = 1.0 - P \left( Z \leq \frac{10 - 0.0}{\sqrt{43.74}} \right) \]
\[ = 1.0 - P(Z \leq 1.51) \]
\[ = 1.0 - \Phi(1.51) \]
\[ = 1.0 - 0.9345 \]
\[ = 0.0655 \]

(e) The probability that 15 fishes selected at random have an average length less than 72 cm:

\[ Y = \bar{X} \]

\[ \mu_Y = \frac{15 \times 70}{15} = 70 \]

\[ \sigma^2_Y = \frac{\sigma^2}{n} = \frac{(5.4)^2}{15} = 1.944 \]

\[ P(Y \leq 72) = P \left( Z \leq \frac{72 - 70}{\sqrt{1.944}} \right) \]
\[= P(Z \leq 1.434)\]
\[= \Phi(1.434)\]
\[= 0.9236\]

**Problem (02):**
Adults salmon fish have lengths that are normally distributed with a mean of \(\mu = 70\) cm and a standard deviation of \(\sigma = 5.4\) cm.

(a) (5 points) Suppose that a restaurant accepts only fish with length from 60 cm to 83 cm. What is the expectation of the proportion of fish that will not be accepted by the restaurant?

(b) (5 points) If a fisherman brings 150 fish to the restaurant, what are the expected number of fish that will not be accepted by the restaurant.

(c) (5 points) For "b", what is the probability that the number of fish rejected by the restaurant is less than four fish?

(Question 3: (15 points) in Final Exam 2010)

**Solution:**

(a) The expectation of the proportion of fish that will not be accepted:

\[\mu = 70\]
\[\sigma = 5.4\]

Range = 60 − 83

\(X = a\) random variable for the fish accepted

\(Y = a\) random variable for the proportion of the fish accepted = \(X / n\)

\(E(Y) = P\)

The probability that the fish caught is inside the range:

\[P(60 \leq X \leq 83) = P\left(\frac{60 - 70}{5.4} \leq Z \leq \frac{83 - 70}{5.4}\right)\]
\[= P(-1.85 \leq Z \leq 2.4)\]
\[= \Phi(2.4) - \Phi(-1.85)\]
\[= 0.9920 - 0.0322\]
\[= 0.9598\]
The probability that the fish caught is outside the range $\approx 1 - 0.9598$

$= 0.0402$

(b) The expected number of fish that will not be accepted:

$P(\text{not accepted}) = 0.0402$

Expected number $= E(X) = np = 150 \times 0.0402 = 6.03$

(c) The probability that the number of fish rejected is less than four:

Since: $np = 6.03 > 5$

$n(1 - p) = 143.9 > 5$

Therefore, the Binomial distribution can be approximated to a Normal distribution with:

$E(X) = \mu = np = 6.03$

$Var(X) = \sigma^2 = np(1 - p) = 150 \times 0.0402 \times (1.0 - 0.0402) = 5.784$

$X \sim N(6.03, 5.784)$

$P(x \leq 4) \approx P(x \leq 4.5)$

$= P \left( Z \leq \frac{4.5 - 6.03}{\sqrt{5.784}} \right)$

$= P(Z \leq -0.6361)$

$= \Phi(-0.6842)$

$= 0.2466$

Problem (03):

A multiple-choice exam has 200 questions, each with four possible answers, of which only one is the correct answer. Suppose that a student guesses on each question.

(a) (5 points) What is the probability that a student will manage to obtain 38 to 40 correct answers?
(b) (5 points) What is the probability that a student will manage to obtain at least 42 correct answers?

(c) (additional) In how many ways can a student check off one answer to each question?

(d) (5 points) In how many ways can a student check off one answer to each question and get all the answers wrong?

(Question 5: (15 points) in Final Exam 2010)

Solution:

(a) The probability that a student will manage to obtain 38 to 40 correct answers:

\[ P = \frac{1}{4} = 0.25 \]

\[ X \sim B(200, 0.25) \]

Since: \( np = 200 \times 0.25 = 50 > 5 \)

\( n(1-p) = 150 > 5 \)

Therefore, the Binomial distribution can be approximated to a Normal distribution with:

\[ \mu = np = 200 \times 0.25 = 50 \]

\[ \sigma^2 = np(1-p) = 200 \times 0.25 \times (1-0.25) = 37.5 \]

\[ P(38 \leq X \leq 40) = P\left(\frac{37.5 - 50}{\sqrt{37.5}} \leq Z \leq \frac{40.5 - 50}{\sqrt{37.5}}\right) \]

\[ = P(-2.04 \leq Z \leq -1.55) \]

\[ = \Phi(-1.55) - \Phi(-2.04) \]

\[ = 0.06615 - 0.0207 \]

\[ = 0.04545 \]

(b) The probability that a student will manage to obtain at least 42 correct answers:

\[ P(X \geq 42) = 1.0 - P(X \leq 42) \]

\[ = 1.0 - P\left(Z \leq \frac{42.5 - 50}{\sqrt{37.5}}\right) \]
\[= 1.0 - P(Z \leq -1.22)\]
\[= 1.0 - \Phi(-1.22)\]
\[= 1.0 - 0.1112\]
\[= 0.8888\]

(c) In how many ways can a student check off one answer to each question:

Number of ways = \(4^{200}\)

(d) In how many ways can a student check off one answer to each question and get all the answers wrong:

Number of ways = \(3^{200}\)

**Problem (04):**
The probability that an electrical equipment will fail in less than 1000 hours of continuous use is 0.25. Estimate that the probability that among 200 such components:

(a) fewer than 45 will fail in less than 1000 hours of continuous use?
(b) between 20 and 100 will fail in less than 1000 hours of continuous use?

(Question 2: in Midterm Exam 2005)

**Solution:**

(a) Fewer than 45 will fail in less than 1000 hours:

\[X \sim B(200, 0.25)\]

Since: \(np = 200 \times 0.25 = 50 > 5\)

\(n(1-p) = 200 \times (1-0.25) = 150 > 5\)

Therefore, the Binomial distribution can be approximated to a Normal distribution with:

\[\mu = np = 200 \times 0.25 = 50\]
\[
\begin{align*}
\sigma^2 &= np(1-p) = 200 \times 0.25 \times (1-0.25) = 37.5 \\
X &\sim B(50, 37.5) \\
P(\times \leq 45) &= P\left(Z \leq \frac{45.5 - 50}{\sqrt{37.5}}\right) \\
&= P(Z \leq -0.73) \\
&= \Phi(-0.73) \\
&= 0.2327
\end{align*}
\]

(b) Between 20 and 100 will fail in less than 1000 hours :
\[
P(20 \leq \times \leq 100) = P\left(\frac{100.5 - 50}{\sqrt{37.5}} \leq Z \leq \frac{19.5 - 50}{\sqrt{37.5}}\right)
\]
\[
= P(-4.48 \leq Z \leq 8.24)
\]
\[
= \Phi(8.24) - \Phi(-4.48)
\]

Problem (05):
The thickness of glass sheets produced by certain process are normally distributed with a mean of \(\mu = 3\) mm and a standard deviation of \(\sigma = 0.12\) mm.

(a) What is the probability that a glass sheet is thinner than 2.3 mm?
(b) What is the value of "c" for which there is a 99% probability that a glass sheet has a thickness within the interval \([3.0 - c, 3.0 + c]\)?
(c) What is the probability that three glass sheets placed one on top of another have a total thickness greater than 9.5 mm?
(d) What is the probability that seven glass sheets have an average thickness less than 3.1 mm?
(e) What is the smallest number of glass sheets required in order for their average thickness to be between 2.95 and 3.05 mm with a probability of at least 99.7%?

(Question 2: in Midterm Exam 2005 / Final Exam 2012)

Solution:

(a) \(P(X \leq 2.3)\):
\[ P(X \leq 2.3) = 1.0 - P\left( Z \leq \frac{2.3 - 3.0}{0.12} \right) \]
\[ = 1.0 - P\left( Z \leq \frac{2.3 - 3.0}{0.12} \right) \]
\[ = 1.0 - \Phi(-5.8333) \]

(b) \( P(3.0 - c \leq X \leq 3.0 + c) = 0.99 \)
\[ c = \sigma \times Z_{0.005} = 0.12 \times 2.5758 = 0.3091 \]

(c) Three glass sheets placed one on top of another have a total thickness greater than 9.50 mm:
\[
Y = X_1 + X_2 + X_3 \\
\mu_Y = a_1 \times \mu_{X1} + a_2 \times \mu_{X2} + a_3 \times \mu_{X3} = 1 \times 3 + 1 \times 3 + 1 \times 3 = 9 \\
\sigma_Y^2 = a_1^2 \times \sigma_{X1}^2 + a_2^2 \times \sigma_{X2}^2 + a_3^2 \times \sigma_{X3}^2 = 1 \times 0.12^2 + 1 \times 0.12^2 + 1 \times 0.12^2 \\
\quad = 0.0432 \\
Y \sim N(9, 0.0432) \\
P(Y \geq 9.5) = 1.0 - P(Y \leq 9.5) \\
\quad = 1.0 - P\left( Z \leq \frac{9.5 - 9}{\sqrt{0.0432}} \right) \\
\quad = 1.0 - P(Z \leq 2.4) \\
\quad = 1.0 - \Phi(2.4) \\
\quad = 1.0 - 0.9918 \\
\quad = 0.0082 \\
\]

(d) Seven glass sheets have an average thickness less than 3.10 mm:
\[ \mu = 3.0 \]
\[ \sigma^2 = \frac{(0.12)^2}{7} = 0.00205 \]
\[ P(X \leq 3.1) = P\left( Z \leq \frac{3.1 - 3}{\sqrt{0.00205}} \right) \]
\[ = \Phi(2.2) \]
\[ = 0.9861 \]
(e) The smallest number of glass sheets required in order for their thickness to be between 2.95 and 3.05 mm with a probability of at least 99.7%:

\[ E(X) = \mu = p = 3.0 \]
\[ V ar(X) = \sigma^2 = \frac{(0.12)^2}{n} = \frac{0.0144}{n} \]
\[ \sigma = \sqrt{V ar(X)} = \sqrt{\frac{(0.12)^2}{n}} = \frac{0.12}{\sqrt{n}} \]
\[ P(2.95 \leq Z \leq 3.05) \geq 0.997 \]
\[ P \left( \frac{2.95 - 3.0}{0.12/\sqrt{n}} \leq Z \leq \frac{3.05 - 3.0}{0.12/\sqrt{n}} \right) \geq 0.997 \]
\[ \phi \left( \frac{3.05 - 3.0}{0.12/\sqrt{n}} \right) - \phi \left( \frac{2.95 - 3.0}{0.12/\sqrt{n}} \right) \geq 0.997 \]
\[ \phi(0.4167\sqrt{n}) - \phi(-0.4167\sqrt{n}) \geq 0.997 \]
\[ \phi(0.4167\sqrt{n}) - [1 - \phi(0.4167\sqrt{n})] \geq 0.997 \]
\[ \phi(0.4167\sqrt{n}) - 1 + \phi(0.4167\sqrt{n}) \geq 0.997 \]
\[ 2\phi(0.4167\sqrt{n}) \geq 1.997 \]
\[ \phi(0.4167\sqrt{n}) \geq \frac{1.997}{2} \]
\[ \phi(0.4167\sqrt{n}) \geq 0.9985 \]
\[ 0.4167\sqrt{n} \geq 2.965 \]
\[ n \geq 51 \]

**Problem (06):**

The probability that cylindrical concrete specimen strength exceeds 2000 N/cm² is 15%. The coefficient of variation of these strengths is 20%. Assuming a normal distribution, calculate:

(a) The mean of the strength.

(b) The probability that the strength will be between 1300 and 1900 N/cm².

(Question 6: (6 points) in Final Exam 2016)
Solution:

(a) The mean of the strength :

\[ CV = \frac{\sigma}{\mu} \times 100 \]
\[ 20 = \frac{\sigma}{\mu} \times 100 \]
\[ \sigma = 0.2 \mu \ldots \ldots Eqn. (1) \]

\[ P(X > 2000) = 1.0 - P(X \leq 2000) \]
\[ 0.15 = 1.0 - P(X \leq 2000) \]
\[ P(X \leq 2000) = 1 - 0.15 \]
\[ P(Z \leq z) = 0.85 \]
\[ z = 1.037 \]
\[ x = \mu + \sigma z \]
\[ 2000 = \mu + \sigma \times 1.037 \ldots \ldots Eqn. (2) \]

Solving equations (1) and (2) results in:
\[ \mu = 1656.4519 \text{ N/cm}^2 \]
\[ \sigma = 331 \text{ N/cm}^2 \]

(b) The probability that the strength will be between 1300 and 1900 :

\[ P(1300 \leq X \leq 1900) = P\left( \frac{1300 - 1656.5}{331} \leq Z \leq \frac{1900 - 1656.5}{331} \right) \]
\[ = P(-1.0769 \leq Z \leq 0.7358) \]
\[ = P(Z \leq 0.7358) - P(Z \leq -1.0769) \]
\[ = \phi(0.7358) - \phi(-1.0769) \]
\[ = 0.7689 - 0.1412 \]
\[ = 0.6277 \]