PhD Program – Water Technology
Groundwater Hydrology
(WTEC 9309)

Topic 2:
Groundwater Movement

Instructor:
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3. Groundwater Movement

- Darcy's Law
- Permeability
- Determination of Hydraulic Conductivity
- Anisotropic Aquifers
- Groundwater Flow Rates
- Groundwater Flow Directions
- Dispersion
- Unsaturated Flow
3.1 **Darcy’s Law**

More than a century ago Henry Darcy, *a French hydraulic engineer, investigated the flow of water through horizontal beds of sand to be used for water filtration. He reported* 24 *in 1856:

This statement, that the flow rate through porous media is proportional to the head loss and inversely proportional to the length of the flow path, is known universally as Darcy’s law. It, more than any other contribution, serves as the basis for present-day knowledge of groundwater flow. Analysis and solution of problems relating to groundwater movement and well hydraulics began after Darcy’s work.

\[
\frac{p_1}{\gamma} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{v_2^2}{2g} + z_2 + h_L
\]

\[
h_L = \left(\frac{p_1}{\gamma} + z_1\right) - \left(\frac{p_2}{\gamma} + z_2\right)
\]
Therefore, the resulting head loss is defined as the potential loss within the sand cylinder, this energy being lost by frictional resistance dissipated as heat energy. It follows that the head loss is independent of the inclination of the cylinder.

Now, Darcy’s measurements showed that the proportionalities $Q \sim h_L$ and $Q \sim 1/L$ exist. Introducing a proportionality constant $K$ leads to the equation

$$Q = -KA \frac{h_L}{L} \quad (3.1.3)$$

Expressed in general terms

$$Q = -KA \frac{dh}{dl} \quad (3.1.4)$$

or simply

$$\nu = \frac{Q}{A} = -K \frac{dh}{dl} \quad (3.1.5)$$
Darcy's Law

or simply

\[ v = \frac{Q}{A} = -K \frac{dh}{dl} \]  

(3.1.5)

where \( v \) is the *Darcy velocity* or *specific discharge*; \( K \) is the *hydraulic conductivity*, a constant that serves as a measure of the permeability of the porous medium; and \( dh/dl \) is the hydraulic gradient. The negative sign indicates that the flow of water is in the direction of decreasing head. Equation 3.1.5 states Darcy’s law in its simplest form, that the flow velocity \( v \) equals the product of the hydraulic conductivity and the hydraulic gradient.
EXAMPLE 3.1.1

A field sample of an unconfined aquifer is packed in a test cylinder (see Figure 3.1.1). The length and the diameter of the cylinder are 50 cm and 6 cm, respectively. The field sample is tested for a period of 3 min under a constant head difference of 16.3 cm. As a result, 45.2 cm$^3$ of water is collected at the outlet. Determine the hydraulic conductivity of the aquifer sample.

SOLUTION

The cross-sectional area of the sample is

$$A = \frac{\pi D^2}{4} = \frac{\pi(0.06 \text{ m})^2}{4} = 0.00283 \text{ m}^2$$

The hydraulic gradient, $dh/dl$, is given by

$$\frac{dh}{dl} = \frac{-16.3 \text{ cm}}{50 \text{ cm}} = -0.326$$

and the average flow rate is

$$Q = \frac{45.2 \text{ cm}^3}{3 \text{ min}} = 15.07 \text{ cm}^3/\text{min} = 0.0217 \text{ m}^3/\text{day}$$

Apply Darcy's law, Equation 3.1.4, to obtain the hydraulic conductivity as

$$Q = -KA \frac{dh}{dl} \rightarrow K = \frac{Q}{A(dh/dl)} = \frac{0.0217 \text{ m}^3/\text{day}}{(0.00283 \text{ m}^2)(-0.326)} = 23.5 \text{ m/day}$$
**Darcy Velocity**

The velocity \( v \) in Equation 3.1.5 is referred to as the *Darcy velocity* because it assumes that flow occurs through the entire cross section of the material without regard to solids and pores. Actually, the flow is limited only to the pore space so that the *average interstitial velocity*

\[
v_a = \frac{Q}{\alpha A}
\]  

(3.1.6)

where \( \alpha \) is the (effective) porosity. This indicates that for a sand with a porosity of 33 percent, \( v_a = 3v \). To define the actual flow velocity, one must consider the microstructure of the rock material. In water flowing through a sand, for example, the pore spaces vary continuously with location within the medium. This means that the actual velocity is nonuniform, involving endless accelerations, decelerations, and changes in direction. Thus, the actual velocity depends on specifying a precise point location within the medium. For naturally occurring geologic materials, the microstructure cannot be specified three-dimensionally; hence, actual velocities can only be quantified statistically.
Validity of Darcy law

In applying Darcy’s law it is important to know the range of validity within which it is applicable. Because velocity in laminar flow, such as water flowing in a capillary tube, is proportional to the first power of the hydraulic gradient (Poiseuille’s law), it seems reasonable to believe that Darcy’s law applies to laminar flow in porous media. For flow in pipes and other large sections, the Reynolds number, which expresses the dimensionless ratio of inertial to viscous forces, serves as a criterion to distinguish between laminar and turbulent flow. Hence, by

\[ N_R = \frac{\rho v D}{\mu} \]  \hspace{1cm} (3.1.7)

Experiments show that Darcy’s law is valid for \( N_R < 1 \) and does not depart seriously up to \( N_R = 10. \) This, then, represents an upper limit to the validity of Darcy’s law. A range of val-

Fortunately, most natural underground flow occurs with \( N_R < 1 \), so Darcy’s law is applicable. Deviations from Darcy’s law can occur where steep hydraulic gradients exist, such as near pumped wells; also, turbulent flow can be found in rocks such as basalt and limestone\(^{17}\) that contain large underground openings.*
3.2 Permeability

Intrinsic Permeability

The permeability of a rock or soil defines its ability to transmit a fluid. This is a property only of the medium and is independent of fluid properties. To avoid confusion with hydraulic conductivity, which includes the properties of groundwater, an intrinsic permeability \( k \) may be expressed as

\[
k = \frac{K \mu}{\rho g}
\]  (3.2.1)

where \( K \) is hydraulic conductivity, \( \mu \) is dynamic viscosity, \( \rho \) is fluid density, and \( g \) is acceleration of gravity. Inserting this in Equation 3.1.5 yields

\[
k = -\frac{\mu v}{\rho g (dh/dl)}
\]  (3.2.2)

which has units of

\[
k = -\frac{(kg/ms)(m/s)}{(kg/m^3)(m/s^2)(m/m)} = m^2
\]  (3.2.3)
Hydraulic Conductivity

For practical work in groundwater hydrology, where water is the prevailing fluid, hydraulic conductivity $K$ is employed. A medium has a unit hydraulic conductivity if it will transmit in unit time a unit volume of groundwater at the prevailing kinematic viscosity* through a cross section of unit area, measured at right angles to the direction of flow, under a unit hydraulic gradient. The units are

\[ K = -\frac{v}{dh/dL} \approx \frac{\text{m/day}}{\text{m/m} \alpha} = \text{m/day} \]  

(3.2.6)

indicating that hydraulic conductivity has units of velocity.

Transmissivity

The term transmissivity $T$ is widely employed in groundwater hydraulics. It may be defined as the rate at which water of prevailing kinematic viscosity is transmitted through a unit width of aquifer under a unit hydraulic gradient. It follows that

\[ T = Kb = (\text{m/day})(\text{m}) = \text{m}^2/\text{day} \]  

(3.2.7)

where $b$ is the saturated thickness of the aquifer.
Example

A leaky confined aquifer is overlain by an aquitard that is also overlain by an unconfined aquifer. The estimated recharge rate from the unconfined aquifer into the confined aquifer is 0.085 m/year. Piezometric head measurements in the confined aquifer show that the average piezometric head in the confined aquifer is 6.8 m below the water table of the unconfined aquifer. If the average thickness of the aquitard is 4.30 m, find the vertical hydraulic conductivity, $K_v$, of the aquitard. What type of material could this possibly be?

Given $v = 0.085$ m/year $= 2.329 \times 10^{-4}$ m/day, Equation 3.2.6 is used to compute the vertical hydraulic conductivity of the aquitard:

$$K = -\frac{v}{\frac{dh}{dl}} = -\frac{2.329 \times 10^{-4} \text{ m/day}}{(6.8 \text{ m}/4.30 \text{ m})} = 1.473 \times 10^{-4} \text{ m/day}$$

From Table 3.2.1, the aquitard is composed of clay.
3.2.4 Hydraulic Conductivity of Geologic Materials

The hydraulic conductivity of a soil or rock depends on a variety of physical factors, including porosity, particle size and distribution, shape of particles, arrangement of particles, and other factors.\textsuperscript{63, 79} In general, for unconsolidated porous media, hydraulic conductivity varies with particle size; clayey materials exhibit low values of hydraulic conductivity, whereas sands and gravels display high values.

![Figure 3.2.1. Hydraulic conductivity of various proportions of two uniform sands (courtesy Illinois State Water Survey).](image-url)
### 3.2.4 Hydraulic Conductivity of Geologic Materials

**Table 3.2.1 Representative Values of Hydraulic Conductivity (after Morris and Johnson\(^75\))**

<table>
<thead>
<tr>
<th>Material</th>
<th>Hydraulic conductivity (m/day)</th>
<th>Type of measurement(^a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravel, coarse</td>
<td>150</td>
<td>R</td>
</tr>
<tr>
<td>Gravel, medium</td>
<td>270</td>
<td>R</td>
</tr>
<tr>
<td>Gravel, fine</td>
<td>450</td>
<td>R</td>
</tr>
<tr>
<td>Sand, coarse</td>
<td>45</td>
<td>R</td>
</tr>
<tr>
<td>Sand, medium</td>
<td>12</td>
<td>R</td>
</tr>
<tr>
<td>Sand, fine</td>
<td>2.5</td>
<td>R</td>
</tr>
<tr>
<td>Silt</td>
<td>0.08</td>
<td>H</td>
</tr>
<tr>
<td>Clay</td>
<td>0.0002</td>
<td>H</td>
</tr>
<tr>
<td>Sandstone, fine-grained</td>
<td>0.2</td>
<td>V</td>
</tr>
<tr>
<td>Sandstone, medium-grained</td>
<td>3.1</td>
<td>V</td>
</tr>
<tr>
<td>Limestone</td>
<td>0.94</td>
<td>V</td>
</tr>
<tr>
<td>Dolomite</td>
<td>0.001</td>
<td>V</td>
</tr>
<tr>
<td>Dune sand</td>
<td>0.08</td>
<td>V</td>
</tr>
<tr>
<td>Loess</td>
<td>5.7</td>
<td>V</td>
</tr>
<tr>
<td>Schist</td>
<td>0.2</td>
<td>V</td>
</tr>
<tr>
<td>Slate</td>
<td>0.00008</td>
<td>V</td>
</tr>
<tr>
<td>Till, predominantly sand</td>
<td>0.49</td>
<td>R</td>
</tr>
<tr>
<td>Till, predominantly gravel</td>
<td>30</td>
<td>R</td>
</tr>
<tr>
<td>Tuff</td>
<td>0.2</td>
<td>V</td>
</tr>
<tr>
<td>Basalt</td>
<td>0.01</td>
<td>V</td>
</tr>
<tr>
<td>Gabbro, weathered</td>
<td>0.2</td>
<td>V</td>
</tr>
<tr>
<td>Granite, weathered</td>
<td>1.4</td>
<td>V</td>
</tr>
</tbody>
</table>

\(^a\)H is horizontal hydraulic conductivity, R is a repacked sample, and V is vertical hydraulic conductivity.
Figure 3.2.2. Range of values of hydraulic conductivity and permeability (Freeze, R. A. and Cherry, J. A., *Groundwater*, Prentice Hall, Englewood Cliffs, NJ, 1979.)
3.3 Determination of Hydraulic Conductivity

Hydraulic conductivity in saturated zones can be determined by a variety of techniques, including calculation from formulas, laboratory methods, tracer tests, auger hole tests, and pumping tests of wells.

3.3.1 Formulas

Numerous investigators have studied the relationship of permeability or hydraulic conductivity to the properties of porous media. Several formulas have resulted based on analytic or experimental work. Most permeability formulas have the general form

\[ k = cd^2 \]  
\[ k = f_s f_\alpha d^2 \]  \hspace{1cm} (3.3.1) \hspace{1cm} (3.3.2)

where \( c \) is a dimensionless coefficient,

where \( f_s \) is a grain (or pore) shape factor, \( f_\alpha \) is a porosity factor, and \( d \) is characteristic grain diameter.\(^{30, 62, 68}\) Few formulas give reliable estimates of results because of the difficulty of including all possible variables in porous media. For an ideal medium, such as an assemblage of spheres of uniform diameter, hydraulic conductivity can be evaluated accurately from known porosity and packing conditions.
3.3.2 Laboratory Methods

In the laboratory, hydraulic conductivity can be determined by a permeameter, in which flow is maintained through a small sample of material while measurements of flow rate and head loss are made.\textsuperscript{107} The constant-head and falling-head types of permeameters are simple to operate and widely employed.

![Permeameter Diagrams](image)

**Figure 3.3.1.** Permeameters for measuring hydraulic conductivity of geologic samples. (a) Constant head. (b) Falling head.
3.3.2 Laboratory Methods

\[ K = \frac{VL}{Ath} \]  

(3.3.3)

where \( V \) is the flow volume in time \( t \), and the other dimensions, \( A, L, \) and \( h \), are shown in Figure 3.3.1a. It is important that the medium be thoroughly saturated to remove entrapped air. Several different heads in a series of tests provide a reliable measurement.
Determination of Hydraulic Conductivity

A second procedure utilizes the falling-head permeameter illustrated in Figure 3.3.1b. Here water is added to the tall tube; it flows upward through the cylindrical sample and is collected as overflow.

The test consists of measuring the rate of fall of the water level in the tube. The hydraulic conductivity can be obtained by noting that the flow rate $Q$ in the tube

$$Q = \pi r_t^2 \frac{dh}{dt}$$

must equal that through the sample, which by Darcy's law is

$$Q = \pi r_c^2 K \frac{h}{L}$$

After equating and integrating,

$$K = \frac{r_t^2 L}{2 r_c^2 t} \ln \frac{h_1}{h_2}$$

where $L$, $r_t$, and $r_c$ are shown in Figure 3.3.1b, and $t$ is the time interval for the water level in the tube to fall from $h_1$ to $h_2$. 
A field sample of medium sand with a median grain size of 0.84 mm will be tested to determine the hydraulic conductivity using a constant-head permeameter. The sample has a length of 30 cm and a diameter of 5 cm.

If the field sample in Example 3.3.1 is tested with a head difference of 5.0 cm and 200 ml of water is collected at the outlet in 15 min, determine the hydraulic conductivity of the sample. What should the maximum allowable piezometric head difference be for a series of tests?

**SOLUTION**

Equation 3.3.3 is used to compute the hydraulic conductivity in a constant-head permeameter test:

\[
K = \frac{V}{A} = \frac{\left(200 \text{ cm}^3\right)(30 \text{ cm})}{\left(\frac{\pi(5 \text{ cm})^2}{4}\right)\left(15 \text{ min} \times 60 \frac{s}{\text{min}}\right)(5.0 \text{ cm})} = 0.0679 \text{ cm/s} = 58.7 \text{ m/day}
\]

Based upon this estimate and referring to Example 3.3.1, the maximum allowable piezometric head difference for tests should be approximately

\[
|dh| \leq \frac{(103.6 \text{ m/day})(0.30 \text{ m})}{58.7 \text{ m/day}} \approx 0.53 \text{ m} = 53 \text{ cm}
\]
A 20-cm long field sample of silty, fine sand with a diameter of 10 cm is tested using a falling-head permeameter. The falling-head tube has a diameter of 3.0 cm and the initial head is 8.0 cm. Over a period of 8 hr, the head in the tube falls to 1.0 cm. Estimate the hydraulic conductivity of the sample.

**SOLUTION**

Equation 3.3.6 is used to compute the hydraulic conductivity in a falling-head permeameter test:

\[
K = \frac{r_i^2 L}{r_c^2 t} \ln \frac{h_1}{h_2} = \frac{(1.5 \text{ cm})^2 (20 \text{ cm})}{(5.0 \text{ cm})^2 (8 \times 3600 \text{ sec})} \ln \frac{8.0 \text{ cm}}{1.0 \text{ cm}} = 1.3 \times 10^{-4} \text{ cm/s} = 0.112 \text{ m/day}
\]
Field determinations of hydraulic conductivity can be made by measuring the time interval for a water tracer to travel between two observation wells or test holes. For the tracer, a dye, such as sodium fluorescein, or a salt, such as calcium chloride, is convenient, inexpensive, easy to detect, and safe. Figure 3.3.2 shows the cross section of a portion of an unconfined aquifer with groundwater flowing from hole A toward hole B. The tracer is injected as a slug in hole A after which samples of water are taken from hole B to determine the time of passage of the tracer. Because the tracer flows through the aquifer with the average interstitial velocity $v_a$, then

$$v_a = \frac{K}{\alpha} \frac{h}{L}$$

$$v_a = \frac{L}{t}$$

$$K = \frac{\alpha L^2}{ht}$$

Figure 3.3.2. Cross section of an unconfined aquifer illustrating a tracer test for determining hydraulic conductivity.
3.3.3 Tracer Tests

Although this procedure is simple in principle, results are only approximations because of serious limitations in the field.

1. The holes need to be close together; otherwise, the travel time interval can be excessively long.

2. Unless the flow direction is accurately known, the tracer may miss the downstream hole entirely. Multiple sampling holes can help, but these add to the cost and complexity of conducting the test.

3. If the aquifer is stratified with layers with differing hydraulic conductivities, the first arrival of the tracer will result in a conductivity considerably larger than the average for the aquifer.
A tracer test is conducted to determine the hydraulic conductivity of an unconfined aquifer. The water levels in the two observation wells 20 m apart are 18.4 m and 17.1 m. The tracer injected in the first well arrives at the second observation well in 167 hours. Compute the hydraulic conductivity of the unconfined aquifer given that the porosity of the formation is 0.25.

**SOLUTION**

Given $\alpha = 0.25$, $L = 20$ m, $h = 18.4$ m $- 17.1$ m $= 1.3$ m, $t = 167$ hours $= 6.96$ days, Equation 3.3.9 is used to compute the hydraulic conductivity of the aquifer:

$$K = \frac{\alpha L^2}{ht} = \frac{(0.25)(20 \text{ m})^2}{(1.3 \text{ m})(6.96 \text{ days})} = 11.1 \text{ m/day}$$
3.3.4 Auger Hole Tests

Figure 5.2.4. Hollow-stem auger drilling. The hollow-stem, continuous-flight auger bores into soft soils, carrying the cuttings upward along the flights. When the desired depth is reached, the plug is removed from the bit and withdrawn from inside the hollow stem. A well point on a casing (1\(\frac{1}{4}\) in or 2 in) can then be inserted to the bottom of the hollow stem and the auger pulled out, leaving the small-diameter monitoring well in place (M. L. Scalf\(^{63}\)).
3.3.4 Auger Hole Tests

The auger hole method involves the measurement of the change in water level after the rapid removal of a volume of water from an unlined cylindrical hole. If the soil is loose, a screen may be necessary to maintain the hole. The method is relatively simple and is most adaptable to shallow water table conditions. The value of $K$ obtained is essentially that for a horizontal direction in the immediate vicinity of the hole.

$$K = \frac{C \ dy}{864 \ dt} \quad (3.3.10)$$

where $dy/dt$ is the measured rate of rise in cm/s and the factor 864 yields $K$ values in m/day. The factor $C$ is a dimensionless constant listed in Table 3.3.1 and governed by the variables shown in Figure 3.3.3.
### 3.3.4 Auger Hole Tests

#### Table 3.3.1 Values of the Factor $C$ for the Auger Hole Test to Determine Hydraulic Conductivity (after Boast and Kirkham\textsuperscript{14})

<table>
<thead>
<tr>
<th>$L_w/r_w$</th>
<th>$y/L_w$</th>
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<th>$0.05$</th>
<th>$0.1$</th>
<th>$0.2$</th>
<th>$0.5$</th>
<th>$1$</th>
<th>$2$</th>
<th>$5$</th>
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3.3.5 Pumping Tests of Wells

The most reliable method for estimating aquifer hydraulic conductivity is by pumping tests of wells. Based on observations of water levels near pumping wells, an integrated $K$ value over a sizable aquifer section can be obtained. Then, too, because the aquifer is not disturbed, the reliability of such determinations is superior to laboratory methods. Pump test methods and computations are described in Chapter 4.
3.4 Anisotropic Aquifers

The discussion of hydraulic conductivity up to this point assumed that the geologic material was homogeneous and isotropic, implying that the value of $K$ was the same in all directions. However, this is rarely the case, particularly for undisturbed unconsolidated alluvial materials. The value of $K$. If the layers are horizontal, any single layer with a relatively low hydraulic conductivity causes vertical flow to be retarded, but horizontal flow can occur easily through any stratum of relatively high hydraulic conductivity. Thus, the typical field situation in alluvial deposits is to find a hydraulic conductivity $K_x$ in the horizontal direction that will be greater than a value $K_z$ in a vertical direction.

$$q_1 = K_1 z_1$$  

(3.4.1)

$$q_x = q_1 + q_2 = i(K_1 z_1 + K_2 z_2)$$  

(3.4.2)

Figure 3.4.1. Diagram of two horizontal strata, each isotropic, with different thicknesses and hydraulic conductivities.
Anisotropic Aquifers

\[ q_1 = K_1 i z_1 \]  \hspace{1cm} (3.4.1)

\[ q_x = q_1 + q_2 = i(K_1 z_1 + K_2 z_2) \]  \hspace{1cm} (3.4.2)

where \( i \) is the hydraulic gradient and \( K_1 \) and \( z_1 \) are as indicated in Figure 3.4.1. Because \( i \) must be the same in each layer for horizontal flow, it follows that the total horizontal flow \( q_x \) is

\[ q_x = K_x i(z_1 + z_2) \]  \hspace{1cm} (3.4.3)

\[ K_x = \frac{K_1 z_1 + K_2 z_2}{z_1 + z_2} \]  \hspace{1cm} (3.4.4)

\[ K_x = \frac{K_1 z_1 + K_2 z_2 + \cdots + K_n z_n}{z_1 + z_2 + \cdots + z_n} \]  \hspace{1cm} (3.4.5)

This defines the equivalent horizontal hydraulic conductivity for a stratified material.

Figure 3.4.1. Diagram of two horizontal strata, each isotropic, with different thicknesses and hydraulic conductivities.
Anisotropic Aquifers

\[ K_z = \frac{z_1 + z_2}{\frac{z_1}{K_1} + \frac{z_2}{K_2}} \]  \hspace{1cm} (3.4.11)

which can be generalized for \( n \) layers as

\[ K_z = \frac{z_1 + z_2 + \cdots + z_n}{\frac{z_1}{K_1} + \frac{z_2}{K_2} + \cdots + \frac{z_n}{K_n}} \]  \hspace{1cm} (3.4.12)

This defines the *equivalent vertical hydraulic conductivity* for a stratified material.

As mentioned earlier, the horizontal hydraulic conductivity in alluvium is normally greater than that in the vertical direction. This observation also follows from the above derivations; thus, if

\[ K_x > K_z \]  \hspace{1cm} (3.4.13)

\[ \frac{K_x}{K_z} \geq 1 \]  \hspace{1cm} (3.4.16)

Ratios of \( K_x/K_z \) usually fall in the range of 2 to 10 for alluvium,
An unconfined aquifer consists of three horizontal layers, each individually isotropic. The top layer has a thickness of 10 m and a hydraulic conductivity of 11.6 m/day. The middle layer has a thickness of 4.4 m and a hydraulic conductivity of 4.5 m/day. The bottom layer has a thickness of 6.2 m and a hydraulic conductivity of 2.2 m/day. Compute the equivalent horizontal and vertical hydraulic conductivities.

Equation 3.4.5 is used to compute the equivalent horizontal hydraulic conductivity:

\[
K_x = \frac{K_1z_1 + K_2z_2 + K_3z_3}{z_1 + z_2 + z_3}
\]

\[
= \frac{(11.6 \text{ m/day})(10 \text{ m}) + (4.5 \text{ m/day})(4.4 \text{ m}) + (2.2 \text{ m/day})(6.2 \text{ m})}{(10 \text{ m} + 4.4 \text{ m} + 6.2 \text{ m})} = 7.25 \text{ m/day}
\]

The equivalent vertical hydraulic conductivity is computed using Equation 3.4.12:

\[
K_z = \frac{z_1 + z_2 + z_3}{\frac{z_1}{K_1} + \frac{z_2}{K_2} + \frac{z_3}{K_3}}
\]

\[
= \frac{10 \text{ m} + 4.4 \text{ m} + 6.2 \text{ m}}{\frac{10 \text{ m}}{11.6 \text{ m/day}} + \frac{4.4 \text{ m}}{4.5 \text{ m/day}} + \frac{6.2 \text{ m}}{2.2 \text{ m/day}}} = 4.42 \text{ m/day}
\]

Note that the equivalent hydraulic conductivities above are computed based on the assumption that each layer is individually isotropic, that is, \(K_r = K_z\) in each layer.
3.5 Groundwater Flow Rates

From Darcy’s law it follows that the rate of groundwater movement is governed by the hydraulic conductivity of an aquifer and the hydraulic gradient. To obtain an idea of the order of magnitude of natural velocities, assume a productive alluvial aquifer with \( K = 75 \) m/day and a hydraulic gradient \( \frac{dh}{dl} = -10 \) m/1000 m = -0.01. Then from Equation 3.1.5,

\[
v = K \frac{dh}{dl} = 75(-0.01) = 0.75 \text{ m/day}
\]  

\[
Q = A v = (50)(1000)(0.75) = 37,500 \text{ m}^3/\text{day}
\]

which, when converted to usual streamflow units, amounts to only 0.43 m\(^3\)/s. Thus, groundwater typically can be conceived of as a massive, slow-moving body of water.
3.6 Groundwater Flow Directions

3.6.1 Flow Nets

For specified boundary conditions flow lines and equipotential lines can be mapped in two dimensions to form a flow net. The two sets of lines form an orthogonal pattern of small squares. In a few simplified cases, the differential equation governing flow can be solved to obtain the flow net. Flow net analysis techniques have been applied in a number of ways for groundwater studies. Hollett and Fenemore used flow net analysis to provide initial

If the flow is divided into \( m \) channels by flow lines, then the total flow

\[
Q = m \cdot q = \frac{Kmh}{n}
\]

\( n \) squares between any two adjacent flow lines, the total head loss \( h \)

Figure 3.6.1. Portion of an orthogonal flow net formed by flow and equipotential lines.
Groundwater Flow Directions

Flow in Relation to Groundwater Contours

Because no flow crosses an impermeable boundary, flow lines must parallel it. Similarly, if no flow crosses the water table of an unconfined aquifer, it becomes a bounding flow surface. The

With only three groundwater elevations known from wells, estimates of local groundwater contours and flow directions can be determined as demonstrated by Figure 3.6.4. From field measurements of static water levels in wells within a basin, a water level contour map can be constructed. Flow lines, sketched perpendicular to contours, show directions of movement. An example appears in Figure 3.6.5.
Flow in Relation to Groundwater Contours

which may be interpreted as indicating that in an area of uniform groundwater flow, areas with wide contour spacings (flat gradients) possess higher hydraulic conductivities than those with narrow spacings (steep gradients). Therefore, in Figure 3.6.5, prospects for a productive well are better near section 2 than 1.

\[
\frac{K_1}{K_2} = \frac{i_2}{i_1}
\]

\[
T = \frac{nQ}{mh}
\]

Figure 3.6.5. Contour map of a groundwater surface showing flow lines.
Three observation wells are installed to determine the direction of groundwater movement and the hydraulic gradient in a regional aquifer. The distance between the wells and the total head at each well are shown in Figure 3.6.7a.
SOLUTION

Step 1: Identify the well with the intermediate water level—Well 1 in this case.

Step 2: Along the straight line between the wells with the highest head and the lowest head, identify the location of the same head of the well from Step 1. Note that this is accomplished by locating the elevation of 32.55 m between Well 2 and Well 3 in the graphical solution.

Step 3: Draw a straight line between the intermediate well from Step 1 and the point identified in Step 2. This is a segment of the equipotential line along which the total head is the same as that in the intermediate well (i.e., equipotential line of 32.55 m head in this case).

Step 4: Draw a line perpendicular to the equipotential line passing through the well with the lowest head. The hydraulic gradient is the slope of that perpendicular line. Also, the direction of the line indicates the direction of groundwater movement. The graphical procedure above is illustrated in Figure 3.6.7b. The hydraulic gradient is then computed as

\[ i = \frac{32.55 \text{ m} - 32.41 \text{ m}}{115.93 \text{ m}} = 0.0012 \]
The average daily discharge from the Patuxent Formation (see Figure 3.6.8) in the Sparrows Point district of Baltimore, Maryland, in 1945 was estimated as $1 \times 10^6$ ft$^3$/day. A flow net of the region is constructed using the available contour lines as shown in Figure 3.6.8. (This example is adapted from Lohman.\textsuperscript{66}) Compute the transmissivity of the regional aquifer.
As shown in the flow net, there are 15 flow channels, hence \( m = 15 \). There are four equipotential drops from the 60-ft contour line to the 20-ft contour line, so \( h = 40 \) ft and \( n = 4 \). Then the overall transmissivity of the district can be computed using Equation 3.6.13:

\[
T = \frac{nQ}{mh} = \frac{(4)(1 \times 10^6 \text{ ft}^3/\text{day})}{(15)(40 \text{ ft})} \approx 6700 \text{ ft}^2/\text{day}
\]
In saturated flow through porous media, velocities vary widely across any single pore, just as in a capillary tube where the velocity distribution in laminar flow is parabolic. In addition, the pores possess different sizes, shapes, and orientations. As a result, when a labeled miscible liquid, referred to as a tracer, is introduced into a flow system, it spreads gradually to occupy an increasing portion of the flow region. This phenomenon is known as dispersion and constitutes a nonsteady, irreversible mixing process by which the tracer disperses within the surrounding water.\textsuperscript{20, 84}

Dispersion is essentially a microscopic phenomenon caused by a combination of molecular diffusion and hydrodynamic mixing occurring with laminar flow through porous media.
The equation for dispersion in homogeneous and isotropic media for the two-dimensional case has the form

\[
\frac{\partial c}{\partial t} = D_L \frac{\partial^2 c}{\partial x^2} + D_T \frac{\partial^2 c}{\partial y^2} - \nu \frac{\partial c}{\partial x}
\]  

(3.7.1)

where \( c \) is the relative tracer concentration \((0 \leq c \leq 1)\), \( D_L \) and \( D_T \) are longitudinal and transverse dispersion coefficients, \( \nu \) is fluid velocity, \( x \) is the coordinate in the direction of flow, \( y \) is the coordinate normal to flow, and \( t \) is time. Dimensions of the dispersion coefficients are \( L^2/T \).

**Figure 3.7.2.** Lateral dispersion of a tracer originating from a point source in a porous medium.

**Figure 3.7.3.** Sketches of tracer distribution resulting from dispersion by flow in porous media. (a) Continuous tracer (b) Single slug of tracer.
3.7.2 Dispersion and Groundwater Hydrology

In groundwater hydrology, dispersion may be encountered whenever two fluids with different characteristics come into contact. Prime examples of this include tracers for evaluating directions and velocities of groundwater flow, introduction of pollutants into the ground (Chapter 8), artificial recharge of water with one quality into an aquifer containing groundwater of another quality (Chapter 13), and intrusion of saline water into freshwater aquifers (Chapter 14). In general, the magnitude of dispersion for uniform sands can be measured in terms of only a few meters over a travel distance of $10^3$ meters.

Figure 3.7.4. Dispersion of two dye streams in a heterogeneous porous medium. Dotted areas indicate bands of much higher permeability (after Skibitzke and Robinson\textsuperscript{90}).
3.10 UNSATURATED FLOW

In groundwater hydrology, unsaturated flow is important for downward vertical flow (natural and artificial recharge), upward vertical flow (evaporation and transpiration), movement of pollutants from ground surface, and horizontal flow in the capillary zone above the water table. A large amount of literature exists on unsaturated flow, most of it contributed by soil scientists. Significant summaries of the subject are available in several references. 

In order to discuss infiltration, we must first consider the various subsurface flow processes shown in Figure 3.10.1. These processes are infiltration of water to become soil moisture, subsurface flow (unsaturated flow) through the soil, and groundwater flow (unconfined saturated flow). Unsaturated flow refers to flow through porous medium when some of the voids are occupied by air. Saturated flow occurs when the voids are filled with water. The water table is the interface between the saturated and unsaturated flow where atmospheric pressure prevails. Saturated flow occurs below the water table and unsaturated flow occurs above the water table.
Flow Through Unsaturated Soils

Consider the control volume (element) in Figure (3.10.3b) for an unsaturated soil which has sides of lengths \( dx, dy, \) and \( dz \) with a volume of \( dx \, dy \, dz \). The volume of water contained in the control volume is \( \theta \, dx \, dy \, dz \) [where \( \theta \) is the moisture content]. Flow through the control volume is defined by the Darcy flux, \( q = Q/A \), which is the volumetric flow rate per unit of soil area. For this derivation, the horizontal fluxes are ignored and only the vertical (\( z \)) direction is considered, with \( z \) positive upward.

The time rate of change of mass stored in the element (control volume) is

\[
- \frac{d}{dt} \int \rho \, dV = \frac{d}{dt} (\rho \theta \, dx \, dy \, dz) = \rho \theta \, dx \, dy \, dz \frac{\partial \theta}{\partial t}
\]

(3.10.2)

Figure 3.10.3. (a) Cross-section through an unsaturated porous medium (b) Control volume for development of the continuity equation in an unsaturated porous medium (from Chow et al.21)
Dividing through by $\rho$ $dx$ $dy$ $dz$ and rearranging results in the following continuity equation for one-dimensional unsteady unsaturated flow in a porous medium in the $z$ direction:

$$\frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial z} = 0$$

(3.10.4)

*Darcy’s law* relates the *Darcy flux*, $q$, to the rate of headloss per unit length of medium. For flow in the vertical direction, the headloss per unit length is the change in total head $\partial h$ over a distance, $\partial z$, that is, $-\frac{\partial h}{\partial z}$, where the negative sign indicates that total head decreases (as a result of friction) in the direction of flow. Darcy’s law for unsaturated flow can be expressed as

$$q = -K_u(\theta)\frac{\partial h}{\partial z}$$

(3.10.5)

where $K_u(\theta)$ is the *hydraulic conductivity*, as a function of the moisture content, $\theta$. This law applies to areas that are large compared with the crosssection of individual pores and grains of the medium. Darcy’s law describes a steady uniform flow of constant velocity with a net force of zero in a fluid element. In unconfined saturated flow, the forces are gravity and friction. For unsaturated flow, the forces are gravity, friction, and the *suction force* that binds water to soil particles through surface tension.
In unsaturated flow the void spaces are only partially filled with water so that water is attracted to the particle surfaces through electrostatic forces between the water molecule polar bonds and the particle surfaces. This in turn draws water up around the particle surfaces, leaving air in the center of the voids. The energy due to the soil suction forces is referred to as the suction head (or matric potential) $\psi$ in unsaturated flow, which varies with moisture content (see subsection 3.11.2). Total head is then the sum of the suction and gravity heads:

$$ h = \psi + z \quad (3.10.6) $$

Note that the velocities are so small that there is no term for velocity head in this expression for total head.

Darcy’s law can now be expressed as

$$ q = -K(\theta) \frac{\partial(\psi + z)}{\partial z} \quad (3.10.7) $$

$$ q = -K(\theta)\left(\frac{\partial\psi}{\partial z} + \frac{\partial z}{\partial z}\right) = -K(\theta)\left(\frac{d\psi}{d\theta} \frac{\partial \theta}{\partial z} + 1\right) = -\left(K(\theta) \frac{d\psi}{d\theta} \frac{\partial \theta}{\partial z} + K(\theta)\right) \quad (3.10.9) $$

The soil water diffusivity $D[L^2/T]$ is defined as

$$ D = K(\theta) \frac{d\psi}{d\theta} \quad (3.10.10) $$
The diffusivity is considered as the ratio of hydraulic conductivity to the water capacity of the soil, \( d \theta / d \psi \). Substituting the above expression for \( D \) into Equation 3.10.9 results in

\[
q = -\left( D \frac{\partial \theta}{\partial z} + K(\theta) \right)
\]

(3.10.11)

Using the continuity equation (3.10.4) for one-dimensional, unsteady, unsaturated flow in a porous medium, we obtain

\[
\frac{\partial \theta}{\partial t} = -\frac{\partial q}{\partial z} = \frac{\partial}{\partial z}\left( D \frac{\partial \theta}{\partial z} + K(\theta) \right)
\]

(3.10.12)

which is a one-dimensional form of Richard's equation. This equation is the governing equation for unsteady unsaturated flow in a porous medium (Richards\^{83}). For a homogeneous soil, \( \partial K / \partial z = 0 \), so \( \partial \theta / \partial t = \partial / \partial z(D \partial \theta / \partial z) \).
EXAMPLE 3.10.1

Determine the flux for a soil in which the unsaturated hydraulic conductivity is expressed as a function of the suction head as \( K = 250(-\psi)^{-2.11} \) in cm/d at depth \( z_1 = 80 \) cm, \( h_1 = -145 \) cm, and \( \psi_1 = -65 \) cm at depth \( z_2 = 100 \) cm, \( h_2 = -160 \) cm, and \( \psi_2 = -60 \) cm.

The flux is determined using Equation 3.10.5. Hydraulic conductivity is computed using an average value of \( \psi = -65 + (60)/2 = -62.5 \) cm. Then \( K_u = 250(-\psi)^{-2.11} = 250 (62.5)^{-2.11} = 0.041 \) cm/d. The flux is then

\[
q = -K_u \left( \frac{h_1 - h_2}{z_1 - z_2} \right) = -0.041 \left[ \frac{-145 - (-160)}{-80 - (-100)} \right] = -0.03 \text{ cm/d}
\]

The flux is negative because the moisture is flowing downward in the soil.
Unsaturated Hydraulic Conductivity

Unsaturated flow in the zone of aeration can be analyzed by Darcy’s law; however, the unsaturated hydraulic conductivity $K_u$ is a function of the water content as well as the negative pressure head (tension).\textsuperscript{3, 14, 38, 100} Because part of the pore space is filled with air, the available cross-sectional area available for water flow is reduced; consequently, $K_u$ is always less than the saturated value $K$.

$$\frac{K_u}{K} = \left(\frac{S_s - S_o}{1 - S_o}\right)^3$$

**Figure 3.10.4.** Ratio of unsaturated to saturated hydraulic conductivity as a function of saturation (after Irmay\textsuperscript{55}).
For hydraulic conductivity and negative pressure, S-shaped relations as indicated in Figure 3.10.5 are generally applicable. These can be approximated by a step function or by

\[
\frac{K_u}{K} = \frac{a}{b} (-h)^n + a
\]

(3.10.14)

where \(a\), \(b\), and \(n\) are constants that vary with particle sizes of unconsolidated material and \(h\) is the pressure head measured in centimeters. It can be seen that when \(h = 0\), which occurs at atmospheric pressure, \(K_u = K\). Orders of magnitude of the constants in Equation 3.10.14 for different soils are as follows:

<table>
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<tr>
<th>Material</th>
<th>(a)</th>
<th>(b)</th>
<th>(n)</th>
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</thead>
<tbody>
<tr>
<td>Medium sands</td>
<td>(5 \times 10^9)</td>
<td>(10^7)</td>
<td>5</td>
</tr>
<tr>
<td>Fine sands, sandy loams</td>
<td>(5 \times 10^6)</td>
<td>(10^5)</td>
<td>3</td>
</tr>
<tr>
<td>Loams and clays</td>
<td>(5 \times 10^3)</td>
<td>(5 \times 10^3)</td>
<td>2</td>
</tr>
</tbody>
</table>
End of Topic 2