Multi-Rate DSP

Applications:

Oversampling, Undersampling, Quadrature Mirror Filters
Multi-Rate DSP

Oversampling
Optimal Sampling vs. Oversampling

- Sampling at Nyquist rate $F_s = 2F_B$
  - Allows perfect reconstruction in principle, but...
  - Pre-sampling anti-aliasing filter must have very steep roll-off:
    - High-order analogue filter: expensive, difficult, imprecise, large phase distortion, ...
- Sampling at $F_s >> 2F_B$ & decimation to $2F_B$
  - Larger separation between images ⇒ easier filtering of aliases (lower-order filter)
    - Cheaper analogue component; easier digital than analogue VLSI filters; greater digital complexity
Oversampling Noise Reduction

- Quantisation step ($b$-bit ADC, range $R$): $Q = \frac{R}{2^b}$

- Noise power density (per unit sampling bandwidth):
  $$p_N = \frac{\sigma_N^2}{F_s/2} = \frac{Q^2}{12} = \frac{Q^2}{6F_s} \quad \text{[W/Hz]}$$

- Total in-band noise power:
  $$P_{in} = \int_0^{F_s} p_N(f) \, df = \frac{Q^2 F_B}{6F_s} \times \frac{F_B}{F_s/2} = \frac{(R/2^b)^2}{12}$$

- Low $P_{in}$ for high $F_s$:
  $$F_s' \gg 2 F_B \Rightarrow P_{in}' \ll \sigma_N^2$$
**Oversampling:**

**Effective Resolution**

- Equivalent $\beta$-bit ADC operating at $F_s=2F_B$ giving same noise power as $b$-bit ADC operating at $F_s>>2F_B$ over same range $R$:

$$\frac{(R/2^b)^2}{12} \times \frac{F_B}{F_s/2} = \frac{(R/2^\beta)^2}{12} \times \frac{F_B}{2F_B/2}$$

i.e.,

$$\beta = b + \frac{\log_2\left(\frac{F_s}{2F_B}\right)}{2} = b + \frac{\log_2(M)}{2}$$

- $M = \text{oversampling ratio, } \beta-b = \text{resolution increase}$
Oversampling Ratio

Example:

\[ M = 10^6: \quad \beta = b + 10, \text{ i.e.,} \]

standard 16-bit ADC @ \( F_s = 2F_B \) has equivalent resolution w.r.t. noise power as an oversampling 6-bit ADC @ \( F_s = 2 \times 10^6 F_B \) or as oversampling 11-bit ADC @ \( F_s = 2000F_B \)

Thus, 0.5 bit length reduction of ADC per doubling of \( M \)
**ΣΔ (Sigma-Delta) Converters**

- High-rate oversampling allows for differential encoding: only 1 bit needed to quantify change \( \Delta \) between input and delayed output of 1-bit ADC, for closely spaced consecutive samples in output of \( \Sigma \Delta \) quantiser (bitmap of sample increments)

- \( M \)th order alias digital LPF eliminates out-of-band quantisation noise (sharp cut-off); heavy comp.; remedy: perform multiplications at later stage (down-rate \( 2F_B \)) instead

- Long word length of digital o/p determines overall oversampling rate
Converters

- **Accumulator (integrator) & quantiser outputs:**

  \[ w(n) = x(n) - y(n-1) + w(n-1) \]

  \[ y(n-1) = w(n-1) + e(n-1) \]

  \[ y(n) = w(n) + e(n) = x(n) + [e(n) - e(n-1)] \]

  \( \Rightarrow \) **noise transfer function:**

  \[ H(z) = 1 - z^{-1} \]

- **Output psdf (one-sided spectrum):**

  \[ p_y(f) = |H(e^{j\omega T_s})|^2 p_N(f) = 4 \sin^2 \left( \frac{\omega T_s}{2} \right) p_N(f) \]
**ΣΔ Oversampling**

- Precise psdf of output noise depends on pdf & spectral characteristics of \( \{x(n)\} \)
- Assume: \( \{e(n)\} \) is random, uncorrelated, white:

\[
\sigma_N^2 = 12 \frac{Q^2}{F_s}
\]

\( \Rightarrow \) psdf of output noise:

\[
p_y(f) = \frac{2\sin^2(\pi f T_s)}{3} \frac{Q^2}{F_s}
\]
Oversampling: Performance Comparison

- For efficient oversampling ($M >> 1$, $f << F_s$):
  \[
p_y(f) \approx \frac{2(\pi f / F_s)^2 Q^2}{3F_s} \frac{2\pi^2 Q^2 f^2}{3F_s^3}
\]

- Output noise power for $\Sigma\Delta$ modulator:
  \[
P_y = \int_0^{F_s} p_y(f) df = \frac{2\pi^2 Q^2 F_B^3}{9F_s^3}
\]

⇒ Improvement over standard oversampling:

\[
10\log_{10}\left(\frac{P_{in}}{P_y}\right) = 10\log_{10}\left(\frac{3M^2}{\pi^2}\right) = [-5.17 + 20\log_{10}(M)] \text{ dB}
\]
Multi-Rate DSP

Undersampling
Nyquist Condition

- Alias-free (sub)sampling of (discrete) function

\[ x_n(t) = x(t) \delta_{T_s}(t) = x(t) \sum_{k=-\infty}^{+\infty} \delta(t - kT_s) = \sum_{k=-\infty}^{+\infty} x(kT_s) \delta(t - kT_s) \]

- Spectrum (Fourier transformation): convolution

\[ X_n(f) = \int_{-\infty}^{+\infty} X(f - \varphi) \left[ \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \delta(\varphi - kF_s) \right] d\varphi \]

\[ = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(f - \varphi) \delta(\varphi - kF_s) d\varphi \]

Thus,

\[ X_n(f) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X(f - kF_s) \]

- No spectral overlap (aliasing, stroboscopy) iff \[ F_s \geq 2F_B \]
Undersampling: Aliasing

(a)

(b)

(c)

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Undersampling: Baseband

Shannon sampling theorem: \( F_s \geq 2F_B \)

- Applies to baseband signals (DC-coupled): \( F_B \) is largest frequency component in signal
  - Motivation: avoid spectral overlap of baseband frequency responses that are periodically continued due to sampling operation
- For bandpass (narrowband modulated) signals (e.g., radio- and optical communications, IF filters, etc.): condition is too conservative: large spectral gaps occur because \( F_c >> |F_B - F_c| \)
Undersampling: Bandpass

- Aliasing of bandpass signal is avoided if baseband can be folded periodically around carrier frequency without causing overlap.
  - Range of permissible sample frequencies:
    \[ \frac{2F_u}{n} \leq F_s \leq \frac{2F_l}{n-1} \]  
    i.e., \[ 1 \leq n \leq \left\lceil \frac{F_u}{F_u - F_l} \right\rceil \]
- Yields additional permissible lower sampling rates for narrowband signals without aliasing.
  - Practically useful (slower computations)
**Undersampling: Applications**

**Example 1:** digitization of analogue FM audio

\[ F_I = 88 \text{ MHz}, \quad F_u = 108 \text{ MHz} \Rightarrow 1 \leq n \leq 5 \quad \text{(nonzero gap)} \]

- **n=1:** classical (≥ Nyquist rate)
- **n=2:** between modulated \( I_u = 1 \) signal and doubled \( I_f = 2 \) signal
- **n=3:** \( D_u = 2/3, \quad I_f = 1 \)
- **n=5:** \( 43.2 \text{ MHz} \leq F_s \leq 44 \text{ MHz} \Rightarrow 86.4 \text{ MHz} \leq 2...2.5F_s \leq 110 \text{ MHz} \)

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Multi-Rate DSP

Quadrature Mirror Filters for Subband Coding
Subband Coding

Problem statement:
- Efficient transmission of realistic speech or video signals
  - Contain most energy at relative low frequencies (time/space)
  - Coding scheme to be tailored to assign more bits to LF band

Solution:
- Subband coding:
  - divide total frequency band in unequal subbands;
    - narrowest subband for interval with highest energy (equalization of power across band)
  - each subband is encoded separately
  - is alternative to companding (pre-distortion)
Example:

- Multi-rate conversion by factor $I/D$ after each frequency subdivision (LPF/HPF)

- Reduced bitrate of digitized signal (bandwidth compression) due to nonuniform coding (variable number of bits per sample)
Implementation:

Brickwall Filter: Physically unrealizable

Quadrature Mirror Filter (QMF): aliasing for decimated subbands can be removed by judicious choice of $H_0(\omega)$ and $H_1(\omega)$
Two-Channel QMF

- Implementation (analyzer / synthesizer):

\[ x(n) \rightarrow H_0(z) \downarrow 2 \rightarrow X_0 \rightarrow \uparrow 2 \rightarrow Y_0 \rightarrow G_0(z) \rightarrow y(n) \]

\[ x(n) \rightarrow H_1(z) \downarrow 2 \rightarrow X_1 \rightarrow \uparrow 2 \rightarrow Y_1 \rightarrow G_1(z) \rightarrow y(n) \]

(for I/D=1/2)
Two-Channel QMF: Analysis

- QMF Analyzer:

\[ X_0(z) = \frac{1}{D} \sum_{k=0}^{D-1} H_0 \left( \exp \left( -\frac{j2\pi}{D} k \right) z^{\frac{1}{D}} \right) X \left( \exp \left( -\frac{j2\pi}{D} k \right) z^{\frac{1}{D}} \right) \]

\[ D = 2 \]

\[ \Rightarrow X_0(\omega) = \frac{1}{2} \left[ H_0 \left( \frac{\omega}{2} \right) X \left( \frac{\omega}{2} \right) + H_0 \left( \frac{\omega}{2} - \pi \right) X \left( \frac{\omega}{2} - \pi \right) \right] \]

\[ X_1(\omega) = \frac{1}{2} \left[ H_1 \left( \frac{\omega}{2} \right) X \left( \frac{\omega}{2} \right) + H_1 \left( \frac{\omega}{2} - \pi \right) X \left( \frac{\omega}{2} - \pi \right) \right] \]
Two-Channel QMF: Synthesis

- **QMF Synthesizer:**

\[ V_0(z) = Y_0(z^I), \quad I = 2 \]

\[ \Rightarrow Y(\omega) = G_0(\omega)Y_0(2\omega) + G_1(\omega)Y_1(2\omega) \]

- **Cascaded QMF analyzer-synthesizer:**

\[ Y_0(\omega) = X_0(\omega), \quad Y_1(\omega) = X_1(\omega) \]

\[ Y(\omega) = \frac{1}{2}[H_0(\omega)G_0(\omega) + H_1(\omega)G_1(\omega)]X(\omega) + \frac{1}{2}[H_0(\omega - \pi)G_0(\omega) + H_1(\omega - \pi)G_1(\omega)]X(\omega - \pi) \]
Elimination of aliasing for any input signal:

\[ H_0(\omega - \pi)G_0(\omega) + H_1(\omega - \pi)G_1(\omega) = 0 \]

e.g. \[ G_0(\omega) = H_1(\omega - \pi), \quad G_1(\omega) = -H_0(\omega - \pi) \]

- results in time-\textit{invariant} filter
- example: alias-free symmetric subband coding
Distortion-free & alias-free reconstruction:

\[ H_0(\omega)G_0(\omega) + H_1(\omega)G_1(\omega) = D \exp(-jk\omega), \quad D = 2 \]

\[ \Leftrightarrow \quad H_0(\omega)H_1(\omega - \pi) - H_1(\omega)H_0(\omega - \pi) = D \exp(-jk\omega) \]

Example: symmetric subband

\[ H_0^2(\omega) - H_0^2(\omega - \pi) = D \exp(-jk\omega) \]

i.e., \[ |H_0^2(\omega) - H_0^2(\omega - \pi)| \] independent of \( \omega \) (all-pass filter), but may exhibit phase distortion!

It can be shown: linear-phase FIR QMF causes amplitude distortion
**M-Channel QMF Bank**

- **M** branches; \( \downarrow M \) in analyzer, \( \uparrow M \) in synthesizer

- **Output \( k \)th analyzer branch (BPF+D):**

\[
X_k(z) = \frac{1}{M} \sum_{m=0}^{M-1} H_k \left( z^{1/M} \exp \left( -j \frac{2\pi m}{M} \right) \right) X \left( z^{1/M} \exp \left( -j \frac{2\pi m}{M} \right) \right), \quad (M = D)
\]

- **Output synthesizer (I+BPF):**

\[
Y(z) = \sum_{k=0}^{M-1} G_k(z) Y_k(z^M)
\]

\[
\Rightarrow Y(z) = \sum_{k=0}^{M-1} G_k(z) \left[ \frac{1}{M} \sum_{m=0}^{M-1} H_k \left( z \exp \left( -j \frac{2\pi m}{M} \right) \right) X \left( z \exp \left( -j \frac{2\pi m}{M} \right) \right) \right]
\]

\[
= \sum_{m=0}^{M-1} L_m(z) X \left( z \exp \left( -j \frac{2\pi m}{M} \right) \right), \quad L_m(z) = \frac{1}{M} \sum_{k=0}^{M-1} H_k \left( z \exp \left( -j \frac{2\pi m}{M} \right) \right) G_k(z)
\]
M-Channel QMF

- **Alias-free QMF:**

\[ Y(\omega) = L_0(\omega) X(\omega) \quad \text{iff} \quad \sum_{m=1}^{M-1} L_m(z) X \left( z \exp \left( -j \frac{2\pi m}{M} \right) \right) = 0, \quad \forall X(z) \]

i.e. \[ L_m(z) = 0, \quad 1 \leq m \leq M - 1 \]

- **Distortion-free & alias-free QMF:**

\[ |L_0(\omega)| \quad \text{independent of} \quad \omega \quad \text{(all-pass filters)} \]