Chapter 2

Array Signal Processing

In this chapter, the theoretical background will be thoroughly covered starting with giving a brief description of array processing, different array geometries and applications and then defining some terms that are needed to understand following topics in the report. Some light will also be shed on array receivers and what they are composed of, before concluding the chapter with an overview of different localisation algorithms while adding more emphasis on MUSIC and Root MUSIC.

2.1 Array Processing

In literature, sensor array signal processing, array signal processing or simply array processing are three terms that refer to the same field of signal processing.

Definition 5 Array processing is the field of signal processing that uses a number of sensors arrayed in a particular geometry in order to estimate a signal’s parameters and extract it in the presence of noise by using its temporal and spatial characteristics.

From this point on, the terms sensor, element and antenna will be used interchangeably to refer to a sensor of the array.

2.1.1 Issues related to array processing

Four issues are of high importance in array processing[8]:

1. Array configuration:

   There are two main components here. The first is the individual element pattern which determines the shape of how it is corresponding power propagates and in turn, how it contributes to the overall array pattern. Initially,
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antennas are assumed to have an isotropic radiation pattern and afterwards, the actual radiation pattern is incorporated.

The second component is the array geometry, i.e., the geometrical shape that the elements are arrayed in. This can be linear, planar or volumetric (3D). Linear arrays can be uniformly, non-uniformly or randomly spaced. Planar arrays are classified according to their boundaries. For example, a circular array have an outer boundary of circular shape with elements placed inside its circumference. Different array geometries are shown in figure 2.1.

2. Temporal and spatial characteristics of the signal:

Signal in time-domain can be either deterministic or random but when talking about spatial characteristics, a signal me be plane-wave signal from a known direction or unknown direction. It may also be spatially spread signal.

3. Temporal and spatial characteristics of the noise/interference:

Statistically independent random noise is introduced in each individual antenna element. A good system design can significantly lower this noise but it will never be zero. Interference can be a source (or many sources) that is operating in the same radiation area and affects the original signal due to its temporal or spatial properties.

4. Objectives of array processing:

Array signal processing is used for many purposes including: detection the presence of a signal in the presence of noise and interference, demodulate it and estimate the information waveform, detection of the information
sequence of a binary communication signal arriving over multipaths, estimation the DOA of multiple plane-wave signals and the construction the spatiotemporal spectrum of a signal in a noisy environment and finally, direction a signal on to a specific transmission direction.

Array processing is applied in a wide range of applications in many aspect of electrical engineering or interdisciplinary research. These applications include [8]:

- Radar was the first area of application when phased array first used in the World War II.
- Radio astronomy which is a system that is used to detect celestial objects and estimate their characteristics.
- Sonar systems where a system receives an acoustic signal in the water and processes it in order to estimate the spatiotemporal characteristics of this signal.
- Communications systems which use array processing in a plenty of applications including satellite communication, where phased arrays are used. 2G and following mobile generations use array processing to improve the system performance and increase its capacity. Array processing and Information theory are both enabler for Multiple Input Multiple Output (MIMO) communication systems[10].
- Direction finding, i.e., to find DOA, whose main purpose it to locate a transmitting source. Direction finding is our application field that we discuss further in this report.
- Other applications also include seismology and tomography.

### 2.1.2 Definitions

1. **Array Geometry:**

**Definition 6** The array geometry consisting of $N$ elements where $N \geq 2$ is described using a 3D Cartesian space. The array geometry matrix $\mathbf{r} \in \mathbb{R}^{3 \times N}$

$$
\mathbf{r} = \begin{bmatrix}
    r_1, & r_2, & \ldots, & r_N
\end{bmatrix} = \begin{bmatrix}
    r_x, & r_y, & r_z
\end{bmatrix}^T
$$

(2.1)
where \( \mathbf{r}_k = \left[ x_k, y_k, z_k \right]^T \) is a 3×1 real vector that represents the Cartesian coordinates of the \( k \)-th element \( \forall k = 1, 2, ..., N \).

For a linear array, all elements are usually placed on the x-axis with \( y_k \) and \( z_k \) are equal to zero \( \forall k = 1, 2, ..., N \). The same applies to a uniform linear array (ULA).

The notation of the azimuth angle \( \theta \) and the elevation angle \( \phi \) will be borrowed from the spherical coordination systems and will be used to describe a unit vector \( \mathbf{u} \) pointing towards the direction \( (\theta, \phi) \). The azimuth angle, \( \theta \), is measured counterclockwise starting from the x-axis on the x-y plane, while the elevation angle, \( \phi \), is measured counterclockwise starting from the z-axis on the y-z plane[9].

\[
\mathbf{u} = \left[ \cos \theta \cos \phi, \sin \theta \cos \phi \sin \phi \right]^T
\]

**Definition 7** We now define the wavenumber vector \( \mathbf{k} \) in the direction \( (\theta, \phi) \) as:

\[
\mathbf{k} = \mathbf{k}(\theta, \phi) = \left\{ \begin{array}{ll}
\frac{2\pi F_c}{c} \mathbf{u} = \frac{2\pi}{\lambda} \mathbf{u} & \text{in meters} \\
\frac{2\pi F_c}{c} \mathbf{u} \times \frac{1}{\pi} \mathbf{u} & \text{in half wavelength}
\end{array} \right.
\]

where \( F_c, c \) and \( \lambda \) denote the frequency, speed and wavelength of the incident wave, respectively [9].

2. Projection and Projection Operator[9]:

It is impossible to graphically represent an N-dimensional observation space. For convenience, the notation in figure 2-2 will be used.

Let \( \mathbb{A} \) be an \( M \times N \) matrix \( (M < N) \) and \( \mathcal{L}[\mathbb{A}] \) denote the linear subspace spanned by the columns of \( \mathbb{A} \).

**Definition 8** If the \( M \) columns of \( \mathbb{A} \) are linearly independent, then the **dimensionality** of \( \mathcal{L}[\mathbb{A}] \) is:

\[
\dim\{\mathcal{L}[\mathbb{A}]\} = M
\]
Figure 2.3: (a) an N-dimensional complex observation space. (b) One dimensional space spanned by the vector $\mathbf{a}$ (c) an M-dimensional space (with $M \geq 2$) spanned by the columns of the matrix $\mathbf{A}$.

**Definition 9** Let the complement subspace of $\mathcal{L}[\mathbf{A}]$ is denoted by $\mathcal{L}[\mathbf{A}]^\perp$ then, its dimensionality will be:

$$\dim\{\mathcal{L}[\mathbf{A}]^\perp\} = N - M$$  \hspace{1cm} (2.5)

**Definition 10** The projection operator onto subspace $\mathcal{L}[\mathbf{A}]$ is defined as:

$$\mathbb{P}_\mathbf{A} = \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$$  \hspace{1cm} (2.6)

- **Notes:**

1. The sum of $\dim\{\mathcal{L}[\mathbf{A}]\}$ and $\dim\{\mathcal{L}[\mathbf{A}]^\perp\}$ is $N$ which is the dimensionality of the observation space $\mathcal{H}$.

2. A vector $\mathbf{x} \in \mathcal{L}[\mathbf{A}]$ can be written as a linear combination of the columns of $\mathbf{A}$.

3. A vector $\mathbf{x} \in \mathcal{H}$ can be projected onto $\mathcal{L}[\mathbf{A}]$ using the projection operator in (2.6). It also can be projected onto $\mathcal{L}[\mathbf{A}]^\perp$ using the projection operator $\mathbb{P}_\mathbf{A}^\perp = \mathbb{I}_N - \mathbb{P}_\mathbf{A}$, where $\mathbb{I}_N$ is the identity matrix ($N \times N$). Figure 2-3 illustrates the projection concept.

4. For a vector $\mathbf{x} \in \mathcal{L}[\mathbf{A}]$, $\mathbb{P}_\mathbf{A} \mathbf{x} = \mathbf{x}$ and $\mathbb{P}_\mathbf{A}^\perp \mathbf{x} = \mathbf{0}_N$.

5. Properties of $\mathbb{P}_\mathbf{A}$:

   - $N \times N$ matrix
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Figure 2.4: $\mathbb{P}_A \mathbf{x}$ is the projection of $\mathbf{x}$ onto $\mathcal{L}[A]$, and $\mathbb{P}_A^\perp \mathbf{x}$ is the projection of $\mathbf{x}$ onto $\mathcal{L}[A]^\perp$.

- $\mathbb{P}_A \mathbb{P}_A = \mathbb{P}_A$.
- $\mathbb{P}_A = \mathbb{P}_A^H$.

3. Wave Propagation Zones[9]:

Consider an array with $N$ elements which are located according to the geometry matrix $\mathbf{r}$ and is operating in the presence of $M$ narrowband isotropic sources with carrier wavelength $\lambda$. The modeling of the received signal depends significantly on the distance between the transmitter and the receiver. Let $\rho_i$ denote the range between the $i$th emitter at the zero-phase reference point array element positioned at $(0,0,0)$ and let $l_a$ be the array aperture given by:

$$l_a = \max_{i,j} \|r_i - r_j\|$$

(2.7)

1. If $\rho_i \approx 2l_a/\lambda$ : the signal is said to be located close to the near-field (Fresnel zone) border and the spherical wave propagation model should be used.

2. If $\rho_i \gg 2l_a/\lambda$ : the signal is said to be located in the far-field (Fraunhofer zone) and the plane wave propagation model should be used.

3. If $\rho_i \gtrsim 2l_a/\lambda$ but $\rho_i$ is not $\gg 2l_a/\lambda$ : the signal is said to be located in the near far-field and the spherical wave propagation model should be used.

For example, if the separation between the leftmost sensor and the rightmost sensor of a ULA is 0.2 m and the carrier frequency is 2.414GHz ($\lambda =$
0.1243m), then the emitter should be located significantly further than \(2l_a/\lambda = 2 \times 0.2/0.1243 = 3.22m\) in order for the plane-wave model to be utilised.

4. Array Manifold and Array Manifold (Response) Vector \([9,11]\):

In this report, the localisation algorithms that will be discussed are based on super-resolution and signal subspace estimation approach rather than the conventional beamforming approach. Signal subspace approach involves finding the intersection between the signal subspace and the array manifold.

Let \(\mathbf{x}(t)\) be the signal arriving at an \(N\)-element array in vector form from \(M\) sources, then:

\[
\mathbf{x}(t) = \begin{bmatrix} x_1(t), x_2(t), \ldots, x_N(t) \end{bmatrix}^T
\]

\[
= \sum_{i=1}^{M} m_i(t) \mathbf{S}_i + \mathbf{n}(t) \quad (2.8)
\]

where \(\mathbf{n}(t) = [n_1(t), n_2(t), \ldots, n_N(t)]^T \in \mathbb{C}^{N \times 1}\) is the baseband additive white Gaussian noise (AWGN) with power of \(\sigma^2\) and, \(\mathbf{S}_i\) \((N \times 1)\) is the array manifold complex vector associated with the \(i\)th source.

**Definition 11** The array manifold (response) vector \(\mathbf{S}\) is defined as \(N\)-dimensional complex vector associated with the \(i\)th source, which represents the response of an array to a unit amplitude plane-wave impinging from the direction \((\theta_i, \phi_i)\).

\[
\mathbf{S}_i \triangleq \mathbf{S}_i(\theta_i, \phi_i) = \gamma(\theta_i, \phi_i) \odot \exp \left( -j \left[ \mathbf{L}_x, \mathbf{L}_y, \mathbf{L}_z \right] \mathbf{k}(\theta_i, \phi_i) \right) \quad (2.9)
\]

where \(\gamma(\theta_i, \phi_i)\) is \((N \times 1)\) complex response (gain and phase) vector for the \(N\) elements of the array, \(\mathbf{k}(\theta_i, \phi_i)\) is the wavenumber and \(\mathbf{r}\) is the geometry matrix.

**Definition 12** The array manifold is defined as the locus of all manifold vectors in the parameter space \(\{\mathbf{S}(\theta) : 0 \leq \theta \leq \pi\}\) and it describes a curve in \(\mathbb{C}^N\).

For the complete proof of equation 2.9, see section 1.6 in [9]. Note that for a calibrated linear array, \(\mathbf{r}_y = \mathbf{r}_z = \mathbf{0}_N\) and equation 2.9 reduces to:

\[
\mathbf{S}(\theta) = \exp \left( -j \pi \mathbf{r}_x \cos \theta \right) \quad (2.10)
\]
2.2 Problem Statement

Consider an array of \( N \)-elements receiving \( M \) narrowband signals \( (M < N) \) transmitted at the same carrier frequency \( F_c \), i.e., the same wavelength \( \lambda \). Equation 2.8 can be rewritten as:

\[
x(t) = S m(t) + n(t)
\]  

(2.11)

where \( S = [S_1(\theta_1), S_2(\theta_2), ..., S_M(\theta_M)] \in \mathbb{C}^{N \times M} \) is the manifold matrix, \( m(t) = [m_1(t), m_2(t), ..., m_M(t)]^T \in \mathbb{C}^{M \times 1} \) is the signal envelope vector and \( n(t) = [n_1(t), n_2(t), ..., n_N(t)]^T \in \mathbb{C}^{N \times 1} \) is AWGN with power of \( \sigma^2 \).

We want to estimate the directions of arrival \( \{\theta_i : 1 \leq i \leq M\} \) of these \( M \) signals by taking \( L \) snapshots from the received signal \( x(t) \).

The theoretical covariance matrix \( \mathbb{R}_{xx} \) can be constructed using:

\[
\mathbb{R}_{xx} \triangleq \mathbb{E}\{x(t)x(t)^H\}
\]  

(2.12)

\[\mathbb{R}_{xx} = S \mathbb{R}_{mm} S^H + \mathbb{R}_{nn} \]  

(2.13)

where \( \mathbb{R}_{mm} \triangleq \mathbb{E}\{m(t)m(t)^H\} \in \mathbb{C}^{M \times M} \) is the sources covariance matrix, and \( \mathbb{R}_{nn} \triangleq \mathbb{E}\{n(t)n(t)^H\} = \sigma^2 \mathbb{I}_N \) is the isotropic AWGN covariance matrix [12].

In practice, the vector signal \( x(t) \) is observed over a finite interval, with a number of snapshots \( L \), rather than an infinity long observation interval, i.e.:

\[
X = [x(t_1), x(t_2), ..., x(t_L)] \in \mathbb{C}^{N \times L}
\]  

(2.14)

\[
= S M + N
\]  

(2.15)

where \( S = [S_1, S_2, ..., S_M] \in \mathbb{C}^{N \times M} \), \( M = [m(t_1), m(t_2), ..., m(t_L)] \in \mathbb{C}^{M \times L} \) and \( N = [n(t_1), n(t_2), ..., n(t_L)] \in \mathbb{C}^{N \times L} \).

We can calculate the estimated covariance matrix model \( \hat{\mathbb{R}}_{xx} = \hat{\mathbb{R}}_{xx} \) by using a multi-dimensional correlator [12].

\[
\hat{\mathbb{R}}_{xx} \triangleq \frac{1}{L} \sum_{i=1}^{L} x(t_i)x(t_i)^H = \frac{1}{L} XX^H
\]  

(2.16)

Figure 2.4 illustrates the two models of \( \mathbb{R}_{xx} \) (adapted from [13]).

We are interested in subspace direction finding methods, and hence, it is crucial to pay attention to the following properties of the \( \mathbb{R}_{xx}[12] \):

1. The rank of \( \mathbb{R}_{mm} \) is equal to \( M \) if the sources are uncorrelated or partially correlated but it will be smaller than \( M \) if they are correlated. The first case is assumed here.

2. If the sources are different, then the columns of \( S \) are independent which implies that \( S \) is of full rank \( M \).
2.3 Detection Problem

This problem is concerned with finding the number of emitting sources located within the range of an array. That is to say, the solution of detection problem is to find the dimension $M$ of the manifold matrix $S[13]$. The case discussed here is when $(M < N)$.

Using eigen decomposition, the received signal covariance matrix $\mathbb{R}_{xx}$ can be rewritten as [12]:

$$
\mathbb{R}_{xx} = \mathbb{S}\mathbb{R}_{mm}\mathbb{S}^H + \mathbb{R}_{nn}
$$

(2.17)

$$
= \sum_{i=1}^{M} \alpha e_i e_i^H + \mathbb{R}_{nn}
$$

(2.18)

$$
= \mathbb{E}\mathbb{D}\mathbb{E}^H
$$

(2.19)

$$
= \mathbb{E}\times\text{diag}(d_i) \times \mathbb{E}^H
$$

(2.20)

where $e_i, 1 \leq i \leq M$ are the eigenvectors.

$$
d_i = \begin{cases} 
\lambda_i + \sigma_i^2 & 1 \leq i \leq M \\
\sigma_i^2 & M < i \leq N
\end{cases},
$$

$\lambda_i, 1 \leq i \leq M$ are the eigenvalues of $\mathbb{R}_{mm}$ and,

$\sigma_i^2, 1 \leq i \leq N$ are the noise power.

This means that $\mathbb{R}_{mm}$ has $M$ non-zero eigenvalues and $N-M$ zero eigenvalues.

Note that in theory, $\sigma_i^2 = \sigma_j^2 \forall i, j$, while in practice $\sigma_i^2 \neq \sigma_j^2 \forall i \neq j$. To estimate the noise power:

$$
\hat{\sigma}^2 = \frac{1}{N-M} \sum_{i=M+1}^{N} \sigma_i^2
$$

(2.21)
The eigenvectors corresponding to the zero eigenvalues satisfy:

\[ \mathbb{S} \mathbb{R}_{mm} \mathbb{S}^H e_i = 0, \quad M < i \leq N \]  

(2.22)

but \( \mathbb{S} \) and \( \mathbb{R}_{xx} \) are full-rank, hence:

\[ \mathbb{S}^H e_i = 0, \quad M < i \leq N \]  

(2.23)

This equation implies that the noise eigenvectors (which constitute the noise subspace) are orthogonal to the manifold vectors of the \( M \) signals. Furthermore, it implies that due to the orthogonality of noise and signals subspaces, the spans of the manifold vectors and the signals eigenvectors are equal [12].

We are interested in estimating \( M \) and to do this, two common information theoretic methods are discussed.

### 2.3.1 Information Theoretic Detection Criteria:

These criteria are based on the assumption of white noise and non-coherent sources [14].

1. **Akaike Information Criterion (AIC)**

   This criterion is based on an information measure by Kullback-Leibler and is described as a function of parameter \( m, 0 < m < M - 1 \) [14]:

   \[ AIC(m) = L(N - m) \ln \Lambda(m) + m(2N - m) \]  

   (2.24)

   \[ \Lambda(m) = \frac{1}{N-m} \sum_{i=m+1}^{N} \lambda_i \left( \prod_{i=m+1}^{N} \lambda_i \right)^{1/(N-m)} \]  

   (2.25)

   where \( N, M, L \) and \( \lambda_i \) are the number of array elements, dimension of signal subspace, number of snapshots and \( i \)th eigenvalue of \( \mathbb{R}_{xx} \), respectively.

2. **Minimum Description Length (MDL)**

   This method is based on Kolmogorov theory of complexity. Using the same parameters used to describe \( AIC(m) \), \( MDL(m) \) is given by [14]:

   \[ MDL(m) = L(N - m) \ln \Lambda(m) + 0.5m(2N - m) \ln(L) \]  

   (2.26)

### 2.3.2 Comparison Between The Two Criteria

Investigating equations 2.25 and 2.26, we see that they are composed of two terms with the first term \( L(N - m) \ln \Lambda(m) \) common in both equations. This term is
inversely proportional to \( m \) [14]. The second term is a penalty function. In both criteria, the penalty function is directly proportional to \( m \). Combining the two terms together means that there should be a value \( m = m_{opt} = M \) that minimises the sum.

Figure 2-5 describes the probability of error versus the SNR for both MDL and AIC. For AIC, it can be inferred that this criterion is under-penalised and for this reason, there is an inevitable probability of error even at very high SNR values. In contrast to AIC, there is a high probability of error with MDL for a wide range of SNR values, while we have zero error when having high SNR. This implies that MDL is over-penalised [14].

## 2.4 Direction Finding and Estimation

### 2.4.1 Overview

Direction finding algorithms can be split down into two categories: classical methods and subspace methods.

1. **Classical methods** are based on beamforming. The concept they use is to form a beam in a specific direction, scan the parameter space, and measure the received power. The highest powers received correspond to the directions where the signals come from [15].

1.a **Delay-and-Sum**: The power measured by this method is given by:

\[
P(\theta) = \mathcal{E} \left| w^H x(t_i) \right|^2 = \mathcal{E} \left| S(\theta)^H x(t_i) \right|^2 = S(\theta)^H \mathbb{R}_{xx} S(\theta) \quad (2.27)
\]

and will have peaks when the weighing vector \( w \) of the beamformer is equal to the DOA manifold vector. The beampattern is similar to the discrete-time Fourier transform (DTFT) of a rectangular window with only 13dB
difference between the main lobe and the highest gain side lobe and hence this method has a bad resolution. To improve this, an increased number of array elements is required. [15]

1.b Minimum Variance Distortionless Response (MVDR) [15]: This method uses the scanning concept the delay-and-sum uses. However, this method tries to suppress the effect of the noise power or interference coming from directions other than the looking direction. This is achieved by maintaining unity gain towards the looking direction. The weight vector ($w$) is given by:

$$w = \frac{R_{xx}^{-1}S(\theta)}{S(\theta)R_{xx}^{-1}S(\theta)}$$

(2.28)

2. **Subspace methods** have super-resolution capability and can be subdivided into signal subspace methods and noise subspace methods.

2.a In signal subspace methods, the noise subspace is discarded and the signal subspace is retained. This would effectively increase the signal-to-noise ratio (SNR). These methods can be either search-based methods or algebraic methods [12]

2.b Noise subspace methods are based on the fact that the manifold vectors are actually lying in the signal subspace and are orthogonal to the noise subspace. These methods are also divided into search-based methods and algebraic methods[12].

Figure 2.7 categorises the direction finding algorithms and gives some examples on the different categories. See chapter 3 in [12] for details. This report is mainly interested in the two algorithms labeled in red, namely, MUSIC and Root MUSIC.

### 2.4.2 Multiple Source Classification (MUSIC)

Multiple Source Classification is one of the most commonly used direction finding subspace methods. It assumes arbitrary geometric locations, arbitrary directional characteristics in noise or interference and arbitrary covariance matrix. MUSIC can be used in order to estimate several parameter including DOA, with asymptotically unbiased manner. [16,17]

In the discussion of equations 2.22 and 2.23, it was shown that the manifold vectors lie in the signals subspace which is orthogonal to the noise subspace. The idea behind the MUSIC algorithm is to search all the possible manifold vectors
Direction Finding Algorithms

Classical

Subspace

noise subspace

signal subspace

search-based

algebraic

search-based

algebraic

Examples:

Delay-and-Sum
MVDR

Pisarenko
MUSIC
Eigenvector
MODE
Minimum norm

Root MUSIC

Correlogram
Minimum Variance
Autoregressive
Subspace Fitting

ESPRIT
LS-ESPRIT
TLS-ESPRIT
GEESE

Figure 2.7: Direction Finding Algorithms Classification
over the parameter space and to find the intersection between the signal subspace and the array manifold. The manifold vectors that satisfy this intersection are the solution of the direction finding problem since they correspond to the incident signals [15,16].

Let $E_n \in \mathbb{C}^{N \times (N-M)}$ be a matrix whose columns are the noise eigenvectors, where $N$ is the number of array sensors and $M$ is the number of incident signals, and let $\mathcal{S}(\theta)$ be a manifold vector of a signal arriving from direction $\theta$. Denote by $\mathcal{E}(\theta)$ the vector resulting from projecting $\mathcal{S}(\theta)$ on to the noise subspace $\mathcal{L}[E_n]$, then the norm-squared of $\mathcal{E}(\theta)$ is [12]:

$$d^2 = \mathcal{E}(\theta)^H \mathcal{E}(\theta)$$  \hspace{1cm} (2.29)

$$= (\mathbb{P}_{E_n} \mathcal{S}(\theta))^H (\mathbb{P}_{E_n} \mathcal{S}(\theta))$$ \hspace{1cm} (2.30)

$$= \mathcal{S}(\theta)^H \mathbb{P}_{E_n}^H \mathbb{P}_{E_n} \mathcal{S}(\theta)$$ \hspace{1cm} (2.31)

using properties 2 and 3 of $\mathbb{P}_{E_n}$ (See definition 10 in subsection 2.1.2):

$$d^2 = \mathcal{S}(\theta)^H \mathbb{P}_{E_n} \mathcal{S}(\theta)$$ \hspace{1cm} (2.32)

$$= \mathcal{S}(\theta)^H \mathbb{E}_n \cdot (\mathbb{E}_n^H \mathbb{E}_n)^{-1} \mathbb{E}_n^H \mathcal{S}(\theta)$$ \hspace{1cm} (2.33)

But $$(\mathbb{E}_n^H \mathbb{E}_n)^{-1} = \mathbb{I}_N$$

$$d^2 = \mathcal{S}(\theta)^H \mathbb{E}_n \mathbb{E}_n^H \mathcal{S}(\theta)$$ \hspace{1cm} (2.34)

Ideally, $d^2 = 0$. However, in practice $d^2 \approx 0$ due to errors in estimating $E_n$ [15].

The MUSIC spectrum is given by [17]:

$$P_{MU}(\theta) = \frac{1}{\mathcal{S}(\theta)^H \mathbb{E}_n \mathbb{E}_n^H \mathcal{S}(\theta)}$$ \hspace{1cm} (2.35)

Figure 2.8 depicts a MUSIC spectrum $P_{MU}(\theta)$ of three signals arriving from 30$^0$, 35$^0$ and 90$^0$. Note that $P_{MU}(\theta)$ will have very high values at the DOA and low values otherwise.

MUSIC resolution threshold is lower than the resolution threshold of the classical direction finding methods. It is a complicated function of three variables: $N$, number of array sensors, $L$, number of snapshots and SNR [16]. The exact formula can be found in the same reference.

Despite the superiority of MUSIC over other techniques, it fails when the sources are fully correlated (coherent sources) and cannot be applied to the received $\mathbb{R}_{xx}$ directly. Instead preprocessing is needed and if the array is ULA, the preprocessing is performed by partitioning the array into subarrays, possibly
overlapping, and then constructing the covariance matrices of these separate arrays and taking the average as the overall covariance matrix. If the array is not ULA, then forward-backward averaging is applied. MUSIC also does not perform well in the case of small number of snapshots and/or low signal-to-noise ratio. An algorithm called Min-Norm treats this case [16,18].

2.4.3 Root MUSIC

The algebraic subspace direction finding algorithm, Root MUSIC was first introduced in 1983 by Barabell who exploited the geometry of ULA in order to reduce the calculation complexity in the search-based methods when numerically searching for the intersection between the array manifold and the signal subspace. He also showed in his paper that Root MUSIC possesses a better resolution threshold as will be discussed later in this subsection. [19]

Consider a uniform linear array with \(N\) elements and interelement spacing, \(d\), the manifold vector of a signal arriving from direction \(\theta\) can be written as [19]:

\[
\mathbf{S}(\theta) = \begin{bmatrix} e^{j2\pi(d/\lambda)\sin(\theta)}, e^{j4\pi(d/\lambda)\sin(\theta)}, \ldots, e^{j2\pi N(d/\lambda)\sin(\theta)} \end{bmatrix}^T
\]  

(2.36)

The MUSIC spectrum in equation (2.35) can be re-written as:

\[
P_{MU}(\theta) = \frac{1}{\mathbf{S}(\theta)^H \mathbb{P}_{\mathbb{E}_{n}} \mathbf{S}(\theta)}
\]

(2.37)

where \(\mathbb{P}_{\mathbb{E}_{n}} = \mathbb{E}_{n} \mathbb{E}_{n}^H\).
The denominator of the $P_{MU}(\theta)$ can be written in double summation format:

$$P_{MU}(\theta)^{-1} = \sum_{m=1}^{N} \sum_{n=1}^{N} e^{-j2\pi m(d/\lambda)\sin(\theta)} P_{(m,n)} e^{j2\pi n(d/\lambda)\sin(\theta)}$$  \hspace{1cm} (2.38)

$$= \sum_{k=-N+1}^{N+1} p_k e^{-j2\pi k(d/\lambda)\sin(\theta)}$$  \hspace{1cm} (2.39)

where $P_{(m,n)}$ is the element in the $m$th row and $n$th column of $P_{E_n}$ and $p_k$ is the sum of the elements in the $k$th diagonal in $P_{E_n}$. [19]

Define the polynomial:

$$D(z) = \sum_{k=-N+1}^{N+1} p_k z^{-k}$$  \hspace{1cm} (2.40)

The roots of $D(z)$ that are close to the unit circle cause the spectrum $P_{MU}(\theta)$ to have its peaks (elements of manifold vector in equation 2.36 have unity magnitude), and hence, this polynomial can be used to find the DOA as follows:

$$z = z_i = |z_i|e^{j\zeta_i}, 1 \leq i \leq M$$  \hspace{1cm} (2.41)

and peaks will occur when:

$$\zeta_i = 2\pi d \lambda \sin(\theta_i), 1 \leq i \leq M$$  \hspace{1cm} (2.42)

i.e., the DOA, $\theta_i$ [19]:

$$\theta_i = \sin^{-1}\left(\frac{\lambda}{2\pi d} \zeta_i\right), 1 \leq i \leq M$$  \hspace{1cm} (2.43)

The above procedure can be summarised in the following steps:

1. From a given $R_{xx}$ matrix and using the eigenvalue decomposition, find $E_n$.
2. Calculate $P_{E_n} = E_n^* E_n^H$.
3. Find the polynomial coefficients by summing the $2N - 1$ diagonals of $P_{E_n}$ to formulate a polynomial of order $2(N - 1)$.
4. Solve the polynomial and find the phases of the $M$ roots closest to the unit circle.
5. Calculate the DOA, $\theta$, from equation (2.43).

The second advantage of Root MUSIC is the better resolution. Figure 2.9 (adapted from [19]) compares the performance of both MUSIC and Root MUSIC for three signals two of which are very close to each other. Root MUSIC is able to separate those two sources while MUSIC failed. This is due to the fact that a small error in the pole radius estimation does not affect significantly the estimation performance but an error in the angle estimation will [16].
2.5 Reception (Beamforming) Problem

So far a description for the detection and estimation problems was provided. After estimating the DOA of the signals, a system should be able to separate the desired signal and reject all the interfering signals.

When the desired signal and the interference occupy the same frequency band, temporal filtering is not suitable for isolating the desired signal, instead, beamformers are used. Beamformers are linear receivers which uses an array of sensors to spatially isolate the desired signal and cancel the interfering signals [12].

Consider figure 2.10 which illustrates the structure of a narrowband beamformer. The input signals $x_i[n]$, $1 \leq i \leq N$ are multiplied by complex weights to construct the received signal $y[n]$.

$$ y[n] = \sum_{k=1}^{N} w_k^* x_k[n] $$

(2.44)

where $w_k$ is the weight of the $k$th element. In vector form, equation 2.44 can be written as:

$$ y[n] = w^H \mathbf{x} $$

(2.45)
where \( \mathbf{x} = [x_1[n], x_2[n], ..., x_N[n]]^T \in \mathbb{C}^{N \times 1} \) and \( \mathbf{w} = [w_1, w_2, ..., w_N]^T \in \mathbb{C}^{N \times 1} \) [22].

The values of this weight vector determines the array pattern and can be chosen according to many algorithms.

2.5.1 Beamformers

1. Delay-and-Sum Beamformer [22]:

   This is a conventional beamformer with the weight vector elements have equal magnitudes, but different phases. These phases are selected to steer the beamform towards a specific direction where the pattern will have a unity power gain.

   This steering operation is similar to the mechanical steering of an array by is done electronically. This beamformer maximises the SNR but fails when the interference is directional. The weight vector is given by:

   \[
   \mathbf{w} = \frac{1}{N} \mathbf{S}_d(\theta) \tag{2.46}
   \]

   where \( N \) is the number of array elements and \( \mathbf{S}_d(\theta) \) is the manifold vector of the desired signal.

2. Wiener-Hopf Beamformer [3]:

   This is a conventional beamformer whose weight vector \( \mathbf{w} \) is given by:

   \[
   \mathbf{w} = \mathbb{R}_x^{-1} \mathbf{S}_d(\theta) \tag{2.47}
   \]

   This beamformer is optimal in that it maximises the signal-to-noise-plus-interference ratio. It also does not require prior knowledge of the interfering signals’ DOA.

3. Maximum Likelihood Beamformer:

   This beamformer tries to place nulls in the direction of interferences while extracting the desired signal with unity gain [22]. It is data independent and requires a prior knowledge of DOA of both the desired signal and the interference [3]. The weight vector of this beamformer is given by:

   \[
   \mathbf{w} = \text{col}_d(\mathbf{S}^\#) \tag{2.48}
   \]

   where \( \text{col}_d \) means the column of \( \mathbf{S}^\# \) corresponding to the desired signal, and \( \mathbf{S}^\# = \mathbf{S} (\mathbf{S}^H \mathbf{S})^{-1} \) is called the pseudo-inverse of \( \mathbf{S} \) [3].
4. **Super-resolution Beamformer** [3]:

As the name implies, the super-resolution beamformer uses the signal subspace approach. This beamformer is data independent and its resolution is not a function of SNR. The weight vector is given by:

$$ w = \frac{P^\perp}{S \Sigma_d S_d^*} $$

where \([S_d(\theta), S_J] = S\). and \(S_J\) is the manifold matrix of the interfering signals.

This super-resolution beamformer is optimal with respect to the signal-to-interference ratio (SIR) criterion. SIR is maximised at the output of the array.

### 2.5.2 Beamformers Example

1. **Wiener-Hopf Beamformer**

   As can be seen in figures 2.11-13 (for \(SNR = 20dB\)), the attenuation in the directions of interferences is not very high. However, the desired signal has the highest gain amongst the arriving signals but the pattern is not sharp enough to avoid attenuating it. This is apparent in the two cases when the desired signal is at 50\(^\circ\) and 55\(^\circ\) (figures 2.11 and 2.12). In the case when the desired signal is at 120\(^\circ\), the beamformer fails to place its nulls towards the two interfering signals and they are not perfectly cancelled this is due to the fact that this is not a super-resolution beamformer. The performance is deteriorated when the SNR is lower (5\(dB\)).

2. **Maximum Likelihood Beamformer**

   From equation 2.48, it is inferred that this beamformer is a function of DOA but not the covariance matrix, i.e., it is not data dependent and hence, it is not a function of SNR. Obviously, from the figures 2.14-16, it performs better than Wiener-Hopf beamformer in that this beamformer places high attenuation sharp nulls in the directions of the interfering signals while the desired signal has a unity gain.

3. **Super-Resolution Beamformer**

   This beamformer is optimum wrt the SIR criterion. Comparing figures 2.17-19 to figures 2.14-16, it can be seen that the super-resolution beamformer places higher attenuation nulls in the interfering direction than the maximum likelihood does. However, it does not guarantee the unity gain.
Figure 2.11: Wiener-Hopf block output when the desired signal DOA = 50.

Figure 2.12: Wiener-Hopf block output when the desired signal DOA = 55.

Figure 2.13: Wiener-Hopf block output when the desired signal DOA = 120.
Figure 2.14: Maximum Likelihood block output when the desired signal DOA = 50.

Figure 2.15: Maximum Likelihood block output when the desired signal DOA = 55.

Figure 2.16: Maximum Likelihood block output when the desired signal DOA = 120.
Figure 2.17: Super-resolution block output when the desired signal DOA = 50.

Figure 2.18: Super-resolution block output when the desired signal DOA = 55.

at the desired signal direction; it is less than 1 in cases of \( \theta = \{50^\circ, 55^\circ\} \) and greater than 1 in the case of \( \theta = 120^\circ \).

It can be noticed that the pattern of the maximum likelihood beamformer is similar to the super-resolution beamformer. This is due to the fact that \( S^\# = S(S^H S)^{-1} \) can be viewed as a projection operator of the received signal, in other words, this maximum likelihood beamformer is a super-resolution beamformer too.

To sum up, in this chapter the beamforming or reception problem of the array signal processing was discussed. Three beamformers were implemented and their performance was tested. The Wiener-Hopf beamformer is SNR dependent and performs badly when SNR is low. The maximum likelihood beamformer is a super-resolution beamformer that is data independent. It maintains a unity gain in the direction of the desired signal. Similar performance was observed for the Super-Resolution beamformer, but with higher attenuation in the DOA of interference.
2.6 Array Gain/Phase Calibration

The MUSIC and Root MUSIC require the antenna array to be fully calibrated beforehand. Otherwise, the performance of the Direction Finding system will lose the super-resolution capabilities due to electrical uncertainties. i.e., gain, phase and mutual coupling, and geometrical uncertainties, i.e., sensors locations [14]. In this report, and as far as the project is concerned, only gain and phase single-pilot calibration will be discussed.

2.6.1 Mathematical Model

Consider an array of \( N \)-sensors receiving a pilot signal with unity power in the presence of thermal, isotropic and uncorrelated noise with power \( \sigma^2 \), then the covariance matrix of the received signal is [20]:

\[
\hat{R}_{xx} = S \cdot S^H + \sigma^2 I_N
\]

(2.50)

where \( S \) is the manifold vector of the pilot source. \( S \) is a mapping of the transmitting source and the array geometry parameter such as gain, phase and geometry and can modelled as:

\[
S = \gamma \odot \psi \odot \rho
\]

(2.51)

where: \( \gamma = [\gamma_1, \gamma_2, \ldots, \gamma_N]^T \), \( \psi = \exp(j\varphi) \) with \( \varphi = [\varphi_1, \varphi_2, \ldots, \varphi_N] \), \( \rho = \exp(-j\mathbf{r}^T \mathbf{k}) \) and \( \mathbf{r} \) and \( \mathbf{k} \) are the array geometry and wavenumber defined in equations 2.1 and 2.3 respectively.