Signals and Systems

Chapter 1: Introduction

- The concepts of signals and systems arise in a wide variety of fields such as science, communication, biomedical, chemical engineering.

- Although the physical nature of signals and systems may be different, they all have common features:
  1. Signals are represented by one or more independent variables.
  2. All signals contain information about a natural phenomena.
  3. Systems respond to particular signals by producing other signals.

Examples of Signals and Systems:

- Input Signal → Systems → Output Signal

- current/ voltage → electric circuit → Responds to current/v

- photo → Camera → Picture

- number → Program → Result
Why we are studying signals and system?

1. Characterizing the system in detail to understand how it will respond to various input. example: analysis the electric circuit in order to quantify its response to different voltage and current.

2. Designing a system that process signals in particular ways. (enhancement and restoration). example: Design a model for the atmospheric effect in order to eliminate it from the images that come from satellite.

3. Design a system that extract specific pieces of information from signals. example: The estimation of heart rate from an electrocardiogram.

and a lot of application ---------

So →

Signals: a set of data or information, function of independent variables.

Systems: is an entity that processes a set of signals (input) to yield another set of signals (output).
1.1: Size of a Signal

Always we need a measure to quantify the size of anything so we will use [signal energy] to quantify it.

\[ E_x = \int_{-\infty}^{\infty} x^2(t) \, dt. \]

Why \( \int_{-\infty}^{\infty} x^2(t) \, dt \)?

- take in account the amplitude and the duration.
- take in account the variation of signal from positive to negative.

This definition can be generalized to a complex valued signal \( x(t) \) as

\[ E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt. \]

1.1-2: Signal Power

We note that the signal energy is not meaningful if the duration \( \rightarrow \infty \) so we need a new thing to define the size of such signal.

When the amplitude \( |x(t)| \) does not \( \rightarrow 0 \) as \( |t| \rightarrow \infty \) the signal energy is infinite, and we use the power of the signal.
Power of the Signals $P_x$.

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) \, dt.$$ 

* Note that $P_x$ is the time average of the energy.

* Generally, $P_x$ exists if the signal is periodic or has a statistical regularity.

* Example $x=1$ does not have energy or power.

* For periodic signal, averaging $x(t)$ over an infinitely large interval is identical to averaging this quantity over one period.

Comments:

* The signal energy does not indicate the actual energy (in the conventional sense) of the signal because the signal energy depends not only on the signal but also on the load. But if we assume, for example, that the load is 1-ohm,

$$E = \int_{-\infty}^{\infty} \sqrt{P_x} \, dt = \int \sqrt{v^2} \, db.$$
thus the measure of energy is therefore indicative of the energy capability of the signal, not the actual energy.

**Example:** \( x(t) = C \cos(wt + \theta) \) find \( P_x \)

\(-\) \( x(t) \) is periodic \( \rightarrow \) power signal

\[ P_x = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} C^2 \cos^2(wt + \theta) \, dt \]

\[ = \lim_{T \to \infty} \frac{C^2}{T} \int_{0}^{T} \frac{1 + \cos(2wt + \theta)}{2} \, dt \]

\[ = \lim_{T \to \infty} \frac{C^2}{T} \left[ \frac{1}{2} t + \frac{1}{2} \sin(2wt + \theta) \right]_{0}^{T} \]

\[ = \lim_{T \to \infty} \frac{C^2}{T} \left[ \frac{T}{2} \right] = \frac{C^2}{2} \]
1.2: Useful operations.

- **Time Shifting** $t \rightarrow t-b$
  
  - $x(b)$
  
  - $x(b-1)$
  
  - $x(b+1)$

  delay

  advanced

- **Time Scaling** $t \rightarrow at.$
  
  - $x(at)$
  
  - $x(\frac{1}{2})$

- **Time Reversal** $t \rightarrow -t$
  
  - reflection about vertical
  
  - reflection about horizontal

*The order of operation:* $x(t) \rightarrow x(at-b)$.

**First Method:**

- Shift $x(t) \rightarrow x(t-b)$
- Scaling $x(b) \rightarrow x(at)$

**Second Method:**

- Shift $x(at) \rightarrow x(a(t-b))$
- Scaling $x(t-b) \rightarrow x(at-b)$

*The best and recommended*
example: find \[ x(-t-3) \].

1. **Shift**

2. **Time Scale**

\[ x(-t-3) \]

(1.3) **Classification of Signals**

- Continuous-time and discrete-time.

CT → has continuous values for continuous time.

DT → has continuous values for discrete time.

(qualify the nature of the signal on horizontal axis).
Analog and digital Signal.

Analog \rightarrow \text{take continuous values.}

digital \rightarrow \text{take discrete values.}

(qualify the nature of the signal on vertical axis).

\begin{align*}
\text{analog / digital} \\
\text{continuous / discrete.}
\end{align*}

\begin{align*}
\text{examples:}
\end{align*}

\begin{align*}
\text{analog, continuous time} \\
\text{digital / continuous.}
\end{align*}

\begin{align*}
\text{analog / discrete} \\
\text{digital / discrete.}
\end{align*}
(1.3.3) Periodic and aperiodic signals.

To be periodic

1. \( x(t) = x(t + T_0) \) for all \( t \).
2. Start from \(-\infty\) and continue forever.

To: fundamental period: the smallest value of \( T_0 \) that satisfies the periodicity condition.

If:

- \( x(t) = 0 \) \( \forall t \rightarrow \) causal signal
- \( x(t) \) = value \( \rightarrow x < \infty \) \( \rightarrow \) everlasting signal
- \( x(t) = 0 \) \( t > 0 \rightarrow \) anti-causal signal

1.3.4: energy and power signal.

Energy signal \( \rightarrow \) has finite energy \( \left[ x(t) \rightarrow 0 \text{ as } |t| \rightarrow \infty \right] \)

Power signal \( \rightarrow \) has finite power (non-zero).

\( P = \frac{E}{T} \), energy signal \( \rightarrow P = 0, E > 0 \)

\( P \), power signal \( E = \infty \) infinite energy

Deterministic and Random signals:

- Has a description (mathematical
  or graphical).
- Cannot be predicted.
1.4: Some useful signal models:

* Unit step function \( u(t) \).

We use it to describe other signals especially the causal signal.

\[
\begin{array}{c|c}
\quad & \quad \\
 t < 0 & 0 \\
 t \geq 0 & 1 \\
\end{array}
\]

**Example:**

\[ x(t) = u(t-1) + 2u(t-2). \]

**Exercise E.1.8:**

\[ (t-1) [u(t-1) - u(t-2)] + u(t-2) - u(t-4). \]
The unit impulse function is defined as:

\[ \delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & \end{cases} \]

\[ \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \]

*Approximation of \( \delta(t) \):*

\[ \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \delta(t) \, dt = \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \frac{1}{\varepsilon} \, dt \]

\[ = \frac{1}{\varepsilon} \left( \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \right) = 1 \]

\[ \int_{-\infty}^{\infty} k \delta(t) \, dt = k \]

*Multiplication of a function by an impulse:*

\[ \phi(t) \ast \delta(t) = \phi(t) \ast \delta(t) \]

In general, \( \phi(t) \delta(t-T) = \phi(T) \delta(t-T) \).
Sifting property (sampling):
\[ \int_{-\infty}^{\infty} \phi(t) \delta(t) \, dt = \phi(0) \int_{-\infty}^{\infty} \delta(t) \, dt = \phi(0). \]

In General:
\[ \int_{-\infty}^{\infty} \phi(t) \delta(t-T) \, dt = \phi(T). \]

Note: \( \delta(t) \) is not an ordinary function because it is zero everywhere except at \( t=0 \) and is undefined at that point. We called it a generalized function.

\[ \frac{d\delta}{dt} = \delta(t). \]

\[ \int_{-\infty}^{\infty} \delta(t) \, dt = 1 = u(t). \]

Ramp function \( r(t) \):
\[ r(t) = t \, u(t). \]
\[ r(t-T_0) = (t-T_0) \, u(t-T_0). \]
Steps for writing the expression of functions.

1. Find changing points where the slope or amplitude change.

2. Change in the slope at \( t = t_0 \) means that a ramp function is exist.
\[
\Delta S = S_{\text{new}} - S_{\text{old}}
\]
\[
\Delta S = r(t - t_0)
\]

3. Change in the amplitude at \( t = t_0 \) means that a step function is exist.
\[
\Delta A = A_{\text{new}} - A_{\text{old}}
\]
\[
\Delta A = u(t - t_0)
\]

Example: Write an expression for \( x(t) \) using basic functions.

First method (window).
\[
x(t) = (t+2)[u(t+2) - u(t+1)] + [u(t+1) - u(t)] - [u(t) - u(t-2)]
\]

\[
= (t+2)u(t+2) - (t+2)u(t+1) + u(t+1) - u(t) - u(t) + u(t-2)
\]

Note that \( (t+2)u(t+1) = r(t+2) \)
\[
= r(t+2) - u(t+1)(t+2-1) - 2u(t) + u(t-2)
\]
\[
= r(t+2) - r(t+1) - 2u(t) + u(t-2)
\]
Second Method:

* at $b = 2$

$$\Delta S = S_{\text{new}} - S_{\text{old}} = 1 - 0 = 1 \rightarrow r(t + 2).$$
$$\Delta A = 0$$

* at $b = 1$

$$\Delta S = S_{\text{new}} - S_{\text{old}} = 0 - 1 = -1 \rightarrow -r(t + 1).$$
$$\Delta A = 0$$

* at $b = 0$

$$\Delta S = 0$$
$$\Delta A = A_{\text{new}} - A_{\text{old}} = -1 - 1 = -2 \rightarrow -2 u(b).$$

* at $b = 2$

$$\Delta S = 0$$
$$\Delta A = A_{\text{new}} - A_{\text{old}} = 0 - 1 = 1 \rightarrow u(t - 2)$$

$$x(b) = r(b + 2) - r(t + 1) - 2u(b) + u(t - 2).$$

Example 2:

* at $b = 0$

$$\Delta S = 1 - 0 = 1 \rightarrow r(b)$$
$$\Delta A = -1 - 0 = -1 \rightarrow -u(b).$$

* at $b = 1$

$$\Delta S = 0 - 1 = -1 \rightarrow -r(t - 1)$$
$$\Delta A = 1 - 0 = 1 \rightarrow u(t - 1)$$

* at $b = 2$

$$\Delta S = 0$$
$$\Delta A = 0 - 1 = -u(t - 2)$$

$$x(b) = r(b) - u(b) - r(t - 1) + u(b) + u(t - 2).$$
\[(t-1)[u(t) - u(t-1)] + u(t+1) - u(t+2)\]
\[(t-1)u(t) - (t-1)u(t-1) + u(t+1) - u(t+2)\]
\[+ u(t) - u(t-1) - u(t+1) + u(t+2) - u(t+3)\]

1.4.3 \text{ The exponential function } e^s.

\[s = \sigma + \omega j \quad \text{complex in general.}\]

\[e^s = e^{\sigma + j\omega} = e^\sigma e^{j\omega} = e^\sigma (\cos(\omega t) + j\sin(\omega t))\]

15: \text{ even and odd functions:}

1. \text{ Even function } x_e(t) \text{ takes equal values of } x(t) \text{ at } t \text{ and } -t.

2. \text{ } x_e \text{ is symmetrical about vertical axis.}

3. \[x_e(-t) = x_e(t)\]

4. \[x_e(t) = -x_e(-t)\]

5. \text{ } x_o \text{ is antisymmetrical about vertical axis.}

\[x(t) = \frac{1}{2} [x(t) + x(-t)] + \frac{1}{2} [x(t) - x(-t)]\]

\[\text{ even } \quad \text{ odd }\]
Examples:

P.1-1.2: Find the energy of the signals.

\[ x(t) = mt + b \]
\[ m = \text{slope } = -1 \quad , \quad b = 0 \]
\[ x(t) = -t \]
\[ \text{Ex} = \int_{-1}^{0} (-t)^2 \, dt = \left[ \frac{t^3}{3} \right]_{-1}^{0} = \frac{1}{3} \]

* Sign change \( \rightarrow \) no effect on energy.
* Time shifting \( \rightarrow \) no effect on energy.
* Multiplied by \( k \) \( \rightarrow \) the energy \( \times \) multiplied by \( k^2 \).

P.1-1.3:

\[ y(t) \]
\[ x(t) \]

\[ \text{Ex} = \int_{0}^{2\pi} \sin^2(t) \, dt = \int \sin^2(t) \, dt = \left[ \frac{t}{2} - \frac{\sin(2t)}{4} \right]_{0}^{2\pi} \]
\[ = (\pi - 0) - 0 = \pi \]

\[ E_y = \int_{0}^{2\pi} 1 \, dt = 2\pi \]

\[ E_{xy} = \int_{0}^{2\pi} (\sin(t) + 1)^2 \, dt = \int_{0}^{2\pi} \sin^2(t) + 1 + 2\sin(t) \, dt \]
\[ = \int_{0}^{2\pi} \sin^2(t) + \int_{0}^{2\pi} 1 + \int_{0}^{2\pi} 2\sin(t) \, dt = \text{Ex} + E_y = \pi + 2\pi = 3\pi \]

\[ E_{xy} = \text{Ex} + E_y \]
(1.1 - 5) @ x(6) = 5 + 10 \cos (100t + \frac{\pi}{3}).

\[
P = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[ 5 + 10 \cos (100t + \frac{\pi}{3}) \right]^2 dt
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} 25 + 100 \cos^2 \left(100t + \frac{\pi}{3}\right) + \int_{0}^{T} 10 \cos \left(100t + \frac{\pi}{3}\right) dt
\]

\[
= 25 + \frac{100}{2} = 75.
\]

(b) \(10 \cos (100t + \frac{\pi}{3}) + 16 \sinh (150t + \frac{\pi}{3})\).

\[
\frac{C_1^2 + C_2^2}{2} \rightarrow \sin \gamma \neq \omega_2
\]

\[
\frac{1}{2} \left(10^2 + 16\frac{1}{2}\right) = 178.
\]

(c) \((10 + 2 \sin 3\theta) \cos 10t\).

\[
= 10 \cos 10t + 2 \sin 3t \cos 10t
\]

\[
= 10 \cos (10t + 2 \times \frac{1}{2} \left[ \sinh (3t-10t) + \sinh (3t+10t) \right])
\]

\[
= 10 \cos 10t + \sinh (-7t) + \sinh (13t).
\]

\[
P = \frac{10^2}{2} + \frac{1^2}{2} + \frac{1}{2} = 51.
\]

(\(e^{j\omega t} = \frac{e^{j\omega t} + e^{-j\omega t}}{2}\))

\[
= \frac{1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t}
\]

\[
= \frac{1}{2} e^{j(\omega + \omega)} + \frac{1}{2} e^{j(-\omega - \omega)}
\]

\[
= \frac{1}{2} e^{j\omega} + \frac{1}{2} e^{-j\omega}
\]

\[
= e^{j\omega} + e^{-j\omega}
\]

\[
= \frac{1}{2} e^{j\omega} + \frac{1}{2} e^{-j\omega} \quad \frac{1}{2} + \frac{1}{2} = \frac{1}{2}.
\]
1.2.2  Find $x(2t-4)$

1. Shift

2. Scale

Find $x(2-t)$.

1. Shift

2. Scale

1.2.3

Find $x_1, x_2, x_3, x_4$ in term of $x(6)$ using time shift and scaling.
\( x(t) = x(t+1) + x(-t+1) \)

\[
\begin{align*}
\ast x(t+1) &= \begin{cases} 
0 & \text{if } 0 < t < 1 \\
1 & \text{if } 1 \leq t < 2 \\
0 & \text{if } 2 \leq t < 3
\end{cases} \\
x(-t+1) &= \begin{cases} 
0 & \text{if } 0 < t < 1 \\
1 & \text{if } 1 \leq t < 2 \\
0 & \text{if } 2 \leq t < 3
\end{cases}
\end{align*}
\]

\( x(t) = x(t-1) + x(-t+1) + x(t) + x(-t) \)

\( x(t) = x(t-2) + x(-t) + 1 \)

\[
\begin{align*}
y &= mx + b \\
y &= (1 + \frac{1}{2})x + 1 \\
y &= \frac{3}{2}x + 1
\end{align*}
\]

\( y_{so} = 0 \Rightarrow t = \frac{3}{2} \)

1. \( x(2\frac{1}{3} t + 1) + x(-\frac{2}{3} t + 1) \)

2. \( x(2t) \)
\[ x(2(t-1.5)) \quad x(a(t-\frac{9a}{2})) \]

\[ x_u(t) = x(2y_3 + 1) + x(-2y_3 + 1) - \frac{1}{3} x(2t-3) \]

\[ -\frac{1}{3} x(-2t-3) . \]
1.4.1: Sketch the following signals.

(a) \( u(t-5) - u(t-7) \)

(b) \( t^2 \left[ u(t-1) - u(t-2) \right] \)

(c) \( (t-4) \left[ u(t-2) - u(t-5) \right] \)

1.4.3: \( \phi(t) \delta(t-T) = \phi(t) \delta(t-T) \).

(a) \( \frac{\sin t}{t^2 + 2} \delta(t) = \frac{\sin t}{t^2 + 2} \bigg|_{t=0} = \frac{\sin(0)}{0+2} \delta(t) = 0 \delta(t) = 0 \)

(b) \( \frac{\sin(2\pi t)}{(2\pi t)^2 + 7} \delta(t) = \frac{\cos(2\pi t)}{2\pi t} \delta(t) = \frac{2}{9} \delta(t) \)

(c) \( \frac{\sin(\pi t/2 + 4)}{t^2 + 4} \delta(t) = \frac{\sin(\pi/2(1-t))}{t^2 + 4} \delta(t) = -\frac{1}{2} \delta(t-1) \)

Note: \( \delta(t-1) = \delta(t+1) \)
\( (1, y-y) \int_0^\infty \phi(t) \delta(t-T) dt = \phi(T). \)

\( (2) \int_0^\infty \delta(t) x(t+\delta t) dt = x(t) \bigg|_{t=0}^{t=\infty} = x(0) = x(t). \)

\( (3) \int_0^\infty x(t) \delta(t+\delta t) dt = x(t) \bigg|_{t=0}^{t=\infty} = x(t). \)

\( (4) \int_0^\infty \delta(t+3) \sin \pi t dt = \sin \pi t \bigg|_{t=-3}^{t=0} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \)

\( (5) \int_0^\infty (t+4) \delta(1-t) dt = t^2 + 4 \bigg|_{1-4}^{t=\infty} = 5 \)

\( (6) \int_0^\infty x(2-t) \delta(3-t) dt = x(2-t) \bigg|_{3-t}^{t=0} = x(2-3) = x(-1) \)

\( (7) \int_0^\infty x(3-t) \delta(3-t) dt = \text{No solution} \)

\( (8) \int_0^\infty x(3-t) \delta(t+3) dt = \text{No solution} \)
(1.5.7) consider \( y(b) = \frac{1}{5} x(-2b - 3) \).

Find the odd portion

\[
y(a) = \frac{1}{5} x(-2b - 3).
\]

\[
y(a) = \frac{y(a) - y(-a)}{2} = \frac{1}{5} x(-2b - 3) - \frac{1}{5} x(2b + 3)
\]

\[
y(a) = \frac{x(-2b - 3) - x(2b + 3)}{10}
\]

(b) Determine and carefully draw \( x(a) \).

\[
y(a) = \frac{1}{5} x(-2t - 3)
\]

\[
5y(a) = x(-2t - 3) \Rightarrow 5y\left(\frac{t + \frac{3}{2}}{2}\right) = x\left(-\frac{2b}{2}\right)
\]

\[
x(a) = 5y\left(\frac{t + \frac{3}{2}}{2}\right).
\]

For periodic functions that consist of multi function.

\[
T = k_1 T_1 \Rightarrow \frac{T_1}{T_2} = \frac{k_2}{k_1} \quad \text{(must be rational)}.
\]

1. \( x(a) = \sin 2t + \sin \frac{3}{2} \pi t \). (Is this periodic?)

\[
p = \frac{2\pi}{\omega} \Rightarrow p_1 = \frac{2\pi}{2} = \pi, \quad p_2 = \frac{2\pi}{2} = 2
\]

\[
\frac{p_1}{p_2} = \frac{\pi}{2} \neq \text{rational} \Rightarrow x(a) \text{ is aperiodic}.
\]
\[ x(t) = e^{j \frac{\pi}{6} t} + e^{j \frac{5\pi}{6} t} \]

\[ \rho_1 = \frac{25}{10} = 2.5 \quad \rho_2 = \frac{25}{5} = 5 \]

\[ \rho_1 = \frac{12/7}{12/5} \quad \text{rational} \Rightarrow \text{aperiodic} \]
1.6: Systems:

- systems make process on signals.

- systems $\rightarrow$ hardware
  $\rightarrow$ software.

- By driving Mathematical equation that relate
  the input to the output we drive a
  Mathematical Model for the system.

- study systems for $\square$ analysis $\square$ Mathematical Model
  $\square$ design.

1.7: Classification of systems:

- linear and nonlinear systems:
  for linear if $x_1 \rightarrow y_1$
  $x_2 \rightarrow y_2$

  then $\rightarrow$
  ① $x_1 + x_2 = y_1 + y_2$ -- additivity
  ② $kx_1 = ky_1$ -- homogeneity

- To be linear the additivity and the homogeneity
  must be excite.
* we can combine the additivity and the homogeneity in one property which called superposition.

* Superposition:

\[ k_1 x_1 + k_2 x_2 \rightarrow k_1 y_1 + k_2 y_2. \]

* Response of a linear system:

  Total Response = zero input Res + zero-state Res.

(1.7.2) Time - Invariant and Time varying system.

* a system is time invariant if the behavior of the system are fixed over time.

  Mathematically any time shift in the input signal result in an identical time shift in the output signal.

* any system with an input-output relationship described by a linear differential equation is a linear time invariant, when the coefficient are constant.
1.7.3: Instantaneous and Dynamic systems:

* Instantaneous (Memoryless): The output at any instant depends on the input signal at the same instant and not any past or future values of signals.

  * Example: Resistive Circuits,

* Dynamic systems: A system whose response at "t" is completely determined by the input signal over the past T second.

  * Example: RC, RLC, LC circuits.

(1.7.4) Causal and non-causal systems:

* A causal system is one for which the output at any instant "t" depends only on the value of the input x(t) for t ≤ t0.

* The value of the present depends on the present and/or the past, not the future.

* In causal system, the output can not start before the input is applied.
noncausal: mean that the system know the future input and acts on this knowledge before the input is applied.

* the concept of non causality is not real just for the system with time as independent variable.

* example of non-causal.

\[ \text{Diagram: } \]

\[ \text{Note: } \text{All Memoryless systems are causal since the output responds only to the current value of the input.} \]

\[ (1.7.5) \text{ Continuous-Time and Discrete-Time systems.} \]
(1.7.6) Analog and digital systems:

analog signal → analog system → analog signal

digital signal → digital system → digital signal

(1.7.7) Invertible and noninvertible systems.

Invertible ⇒ we can obtain the input signal \( x(t) \) from the output signal \( y(t) \).

non-invertible ⇒ input signal cannot be obtained from the output signal.

example: Rectifier.

(1.7.8) Stable and unstable systems:

bounded input → stable system → bounded output.
Example and notes:

* Steps for checking linearity:

1. Find the response of \( x_1 \) and \( x_2 \):
   \[ x_1 \rightarrow y_1 \quad \text{and} \quad x_2 \rightarrow y_2. \]

2. Find the response of \( x_3 = ax_1 + bx_2 \rightarrow y_3 \).

3. Check if \( y_3 = ay_1 + by_2 \rightarrow \) if so then the system is linear.

Example 1: \( y(\tau) = \frac{dx}{dt} + x(t) \).

(1) \( x_1(\tau) \rightarrow \frac{dx_1}{dt} + x_1(\tau) = y_1(\tau) \).

(2) \( x_2(\tau) \rightarrow \frac{dx_2}{dt} + x_2(\tau) = y_2(\tau) \).

2. For \( x_3 = ax_1(\tau) + bx_2(\tau) \):

\[
\frac{d(ax_1 + bx_2)}{dt} + ax_1 + bx_2 = y_3
\]

\[
a\frac{dx_1}{dt} + ax_1 + b\frac{dx_2}{dt} + bx_2 = y_3
\]

\[
a\left(\frac{dx_1}{dt} + x_1\right) + b\left(\frac{dx_2}{dt} + x_2\right) = y_3
\]

\[
ay_1 + by_2 = y_3
\]

\[\Rightarrow \text{linear.} \]
Check Time invariant:

1. Let \( x(t+\tau) \) be an arbitrary shifted input.

2. If the output equal \( y(t-\tau) \) then time invariant.

Example:

\[
\frac{dy(t)}{dt} = \frac{dx(t+\tau)}{dt} + x(t-\tau) + \frac{dx(t)}{dt}
\]

\( y(t) = \frac{dx(t+\tau)}{dt} + x(t-\tau) \) \( = \) \( \frac{dx(t)}{dt} \)

\( 1 \Rightarrow 2 \Rightarrow \text{time invariant.} \)

Has Memory, Causal.

\( (1.7.1) \)

\[
\frac{dy(t)}{dt} + 2y(t) = x(t)
\]

1. \( x(t) \Rightarrow \frac{dy_1(t)}{dt} + 2y_1(t) = x_1(t) \)

\[
\frac{dy_2(t)}{dt} + 2y_2(t) = x_2(t) + 1
\]

2. Let \( x_3 = ax_1 + bx_2 \Rightarrow \text{non-linear} \)

\[
\frac{dy_3(t)}{dt} + 2y_3(t) = (ax_1 + bx_2)^2 = a^2x_1^2 + b^2x_2^2 + 2abx_1x_2
\]

\( \neq ay_1 + by_2 \)

With Memory / Time invariant / causal
Check time invariant.

1. \( \frac{dy}{dt} + 3y(t) = t^2 x(t) \)

2. \( \frac{dy}{dt} + 3y(t) = t^2 x(t) \)

Let \( x(t) \) be any continuous function

Initial conditions:

1. For \( x_2 = ax + bx^2 \)

\[ \frac{dx_2}{dt} + 3b \frac{dy}{dt} = t^2 (ax + bx^2) \]

\[ x_2 = ax^2 + bx^2 \]

\[ \frac{dx_2}{dt} + 3b \frac{dy}{dt} = t^2 (ax + bx^2) \]

\[ = 2b ax^2 + b \frac{dy}{dt} \]

\[ \Rightarrow \text{linear, complete} \]

---

1. \( \frac{dy}{dt} + 3y(t) = t^2 x(t) \) — (1)

2. \( \frac{dy}{dt} + 3(t-\tau) y(t-\tau) = (t-\tau)^2 x(t-\tau) \) — (2)

\( \Rightarrow 2^{nd} \text{ time variant} \)

with memory / cancel.
\[ y(b) = t \cdot x(t + 10) \]

**Linearity** \[\Rightarrow \quad \text{for } x_3 = ax_1 + bx_2 \]

\[ y_3(b) = t \left( ax_1(t+10) + bx_2(t+10) \right) \]

\[ = at \cdot x_1(t+10) + bt \cdot x_2(t+10) \]

\[ = ay_1 + by_2 \Rightarrow \text{linear}. \]

**Time variant test** \[\Rightarrow \quad y(b) = t \cdot x(t+10-t_0) - - - (1) \]

\[ y(t-t_0) = (t-t_0) \cdot x(t+10-t_0) - - - (2) \]

\[ (1) \neq (2) \quad \text{time variant}. \]

**Causality** \[\Rightarrow \]

non causal since \( y(b) \big|_{t=t_0} = t_0 \cdot x(b_0+10) \)

\( t=t_0 \) future.

\[(1.7.2) \quad y(b) = x(ab). \]

\[ \circ \quad \text{linear.} \]

\[ \circ \quad y = x(a(b-t_0)) - - - (1) \]

\[ y(t-t_0) = x(a(t-t_0)) - - - (2) \]

\[ 1 \neq 2 \Rightarrow \text{time variant}. \]
\( y(u) = x(au) \)

\( x(t + \tau_0) \)  \( \Rightarrow \)  \( x(at - \tau_0) \)

Shift to \( \frac{t}{a} \)

\( y(u) = x(au) \)  \( \Rightarrow \)  \( x(a(t + \tau_0)) \)

\( \tau_0 \)
لا حماية في الـثبيلة بحرية،

أنذروت 1982، مئات السفن من مختلف.

non linear — input

non linear — attraversa input or disturbances

input "function of" t or "function of" t

time variant

time variant

باذل الناتج 8، داخل (8) نطاق كبير بأي حامل 

x(x(t))

 Rentals non causal

with Memory

x(-t)

 Rentals non causal

x(t+a)

 Rentals non causal

n تنبئ السفينة بالمحيط وطائرات

x(-t) = x(0) ⇒ non causal

t=1

x(\pm t) = x(-\pm t) ⇒ non causal.

t=0, -\frac{1}{2}