Properties of DFT

Periodicity and Linearity

1) Periodicity
If $x(n)$ and $X(k)$ are an $N$-point DFT pair, then

$$x(n + N) = x(n) \text{ for all } n$$

$$X(k + N) = X(k) \text{ for all } k$$

2) Linearity

If $x_1(n) \xrightarrow{\text{DFT}} X_1(k)$ and $x_2(n) \xrightarrow{\text{DFT}} X_2(k)$

Then $a_1x_1(n) + a_2x_2(n) \xrightarrow{\text{DFT}} a_1X_1(k) + a_2X_2(k)$
**Symmetry Property**

Let us assume that \( x(n) \) is a real signal and \( X(k) \) is expressed as

\[
X(k) = X_e(k) + jX_i(k) \quad 0 \leq n \leq N - 1
\]

Then the DFT can be written as

\[
X_e(k) = \sum_{n=0}^{N-1} x(n) \cos \left( \frac{2\pi kn}{N} \right) \\
X_i(k) = -\sum_{n=0}^{N-1} x(n) \sin \left( \frac{2\pi kn}{N} \right)
\]

The IDFT can be written as

\[
x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_e(k) \cos \left( \frac{2\pi kn}{N} \right) - X_i(k) \sin \left( \frac{2\pi kn}{N} \right)
\]

---

**3-Symmetry Property**

Let us assume that \( x(n) \) is a *real and even* signal

Then the DFT can be written as

\[
X_e(k) = \sum_{n=0}^{N-1} x(n) \cos \left( \frac{2\pi kn}{N} \right) \\
X_i(k) = 0
\]

The IDFT can be written as

\[
x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_e(k) \cos \left( \frac{2\pi kn}{N} \right)
\]


**Symmetry Property**

Let us assume that \( x(n) \) is a real and odd signal.

Then the DFT can be written as

\[
X_s(k) = 0
\]

\[
X_i(k) = -\sum_{n=0}^{N-1} x(n) \sin \left( \frac{2\pi kn}{N} \right)
\]

The IDFT can be written as

\[
x(n) = -\frac{1}{N} \sum_{k=0}^{N-1} X_i(k) \sin \left( \frac{2\pi kn}{N} \right)
\]

---

**Circular Convolution**

**Circular convolution in the time domain**

If \( x_1(m) = x_1(n) \odot x_2(n) \)

\[
X_s(k) = X_1(k)X_2(k)
\]

**Circular convolution in the frequency domain**

If \( x_1(m) = x_1(n)x_2(n) \)

\[
X_s(k) = \frac{1}{N} X_1(k) \odot X_2(k)
\]
Circular Convolution

Example: Perform the circular convolution of the following two sequences
\[ x_1(n) = \{2,1,2,1\} \text{ and } x_2(n) = \{1,2,3,4\} \]

Solution

The DFT for the first sequence
\[ X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j \frac{2\pi nk}{N}} \]
\[ X_1(k) = 2 + e^{-j\pi/2} + 2e^{-j\pi} + e^{-j3\pi/2} \]
\[ X_1(0) = 6 \quad X_1(1) = 0 \quad X_1(2) = 2 \quad X_1(3) = 0 \]

The DFT for the second sequence
\[ X_2(k) = \sum_{n=0}^{N-1} x_2(n) e^{-j \frac{2\pi nk}{N}} \]
\[ X_2(k) = 1 + 2e^{-j\pi/2} + 3e^{-j\pi} + 4e^{-j3\pi/2} \]

The DFT of the solution is
\[ X_3(k) = X_1(k)X_2(k) \]
\[ X_3(0) = 10 \quad X_3(1) = -2 + 2j \quad X_3(2) = -2 \quad X_3(3) = -2 - 2j \]

Then
\[ X_3(k) = X_1(k)X_2(k) \]
\[ X_3(0) = 60 \quad X_3(1) = 0 \quad X_3(2) = -4 \quad X_3(3) = 0 \]

Now the IDF of \( X_3(k) \) is
\[ x_3(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k) e^{j \frac{2\pi nk}{N}} = \frac{1}{4} (60 - 4e^{j\pi}) \]
\[ x_3(0) = 14 \quad x_3(1) = 16 \quad x_3(2) = 14 \quad x_3(3) = 16 \]
**Circular Shift**

1) Time Circular Shift
   If \( x(n) \leftarrow \text{DFT} \rightarrow X(k) \)
   Then \( x((n-l) \mod N) \leftarrow \text{DFT} \rightarrow X(k)e^{-j2\pi l/N} \)

2) Frequency Circular Shift
   If \( x(n) \leftarrow \text{DFT} \rightarrow X(k) \)
   Then \( x(n)e^{j2\pi l/N} \leftarrow \text{DFT} \rightarrow X((k-l) \mod N) \)

**Circular Correlation**

If \( x(n) \leftarrow \text{DFT} \rightarrow X(k) \) and \( y(n) \leftarrow \text{DFT} \rightarrow Y(k) \)
Then \( r_y(l) \leftarrow \text{DFT} \rightarrow R_y(k) = X(k)Y^*(k) \)

**Linear Filtering**

Use of DFT in linear Filtering

Assume we have a finite duration \( x(n) \) of Length \( L \) as an input to a system with Finite Impulse Response (FIR) of length \( M \)

\[
x(n) = 0 \quad n < 0 \text{ and } n \geq L
\]
\[
h(n) = 0 \quad n < 0 \text{ and } n \geq M
\]

The output sequence

\[
y(n) = x(n) * h(n) \quad \text{or} \quad Y(\omega) = X(\omega)H(\omega)
\]

\( y(n) \) has a finite length which equals \( N = L+M-1 \)
which mean that we can calculate the DFT as

\[
Y(k) = X(\omega) \big|_{\omega=2\pi k/N} \quad k = 0,1,\ldots, N-1
\]
\[
= X(\omega)H(\omega) \big|_{\omega=2\pi k/N} \quad k = 0,1,\ldots, N-1
\]
\[
Y(k) = X(k)H(k) \quad k = 0,1,\ldots, N-1
\]
Linear Filtering

This means that we need to pad each of \( x(n) \) and \( h(n) \) by zeros to make them of length \( N \), this won’t change the frequency response for them as \( N > M \) and \( N > L \).

**Example:** By means of DFT and IDFT, determine the frequency response of the FIR filter with impulse response and input as

\[
h(n) = \{1, 2, 3\} \quad x(n) = \{1, 2, 2, 1\}
\]

**Solution**

\( L = 4 \) and \( M = 3 \) which means that \( L \cdot M - 1 = 6 \)

This means that we need to take DFT of \( N = 6 \) at the least. For simplicity of calculating DFT of size of power 2 as we shall see, we take \( N = 8 \)

The DFT of \( x(n) \) is

\[
X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = 1 + 2e^{-j\pi/4} + 2e^{-j\pi/2} + e^{-j3\pi/4}
\]

\[
X(0) = 6 \quad X(1) = 1.707 - j4.12132
X(2) = -1 - j \quad X(3) = 0.2929 - j0.12132
X(4) = 0 \quad X(5) = 0.2929 + j0.12132
X(6) = -1 + j \quad X(7) = 1.707 + j4.12132
\]

The DFT of \( h(n) \) is

\[
H(k) = \sum_{n=0}^{N-1} h(n) e^{-j2\pi kn/N} = 1 + 2e^{-j\pi/4} + 2e^{-j\pi/2}
\]
Linear Filtering

\[
\begin{align*}
H(0) &= 6 & H(1) &= 2.4142 - j4.4142 \\
H(2) &= -2 - j2 & H(3) &= -0.4142 + j1.5858 \\
H(4) &= 2 & H(5) &= -0.4142 - j1.5858 \\
H(6) &= -2 + j2 & H(7) &= 2.4142 + j4.4142
\end{align*}
\]

The product of these two sequences yields \(Y(k)\)

\[
\begin{align*}
Y(0) &= 36 & Y(1) &= -14.07 - j17.48 \\
Y(2) &= j4 & Y(3) &= 0.07 + j0.515 \\
Y(4) &= 0 & Y(5) &= 0.07 - j0.515 \\
Y(6) &= -j4 & Y(7) &= -14.07 + j17.48
\end{align*}
\]

Finally, the eight point IDFT is

\[
y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{2\pi i n k / N}, \quad n = 0, 1, \ldots, 7
\]

Calculating for each value of \(n\)

\[
y(n) = (1, 4, 9, 11, 8, 3, 0, 0)
\]
Self Study

Students are encouraged to solve the following questions from the textbook:

5.4, 5.5, 5.11, and 5.13.