Structures for FIR Systems

A FIR system has the impulse response

\[ h(n) = \begin{cases} b_n, & 0 \leq n \leq M - 1 \\ 0 & \text{otherwise} \end{cases} \]

Then the output has the form

\[ y(n) = \sum_{k=0}^{M-1} b_k x(n - k) \]

The z-transform for the impulse response is

\[ H(z) = \sum_{k=0}^{M-1} b_k z^{-k} \]

There are different structures to implement FIR system.
Direct Form Structure

\[ y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \]

\[ y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots + h(M-1)x(n-(M-1)) \]

Direct Form Structure

For a linear phase filter

\[ h(n) = \pm h(M - 1 - n) \]

\( M \) is odd
2) Cascade-Form Structure

\[ x(n) = x_1(n) \quad H_1(z) \quad y_1(n) = x_2(n) \quad H_2(z) \quad y_2(n) = x_3(n) \quad H_3(z) \]

\[ H(z) = \prod_{k=1}^{K} H_k(z) \]

\[ H_k(z) = b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}, \quad k = 1, 2, ..., K \]

\[ y_k(n) = y(n) \]

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**Cascade Form Structures**

In case of linear-phase FIR filter, if \( z_k \) and \( (z_k)^* \) are two complex conjugate zeros then \( 1/z_k \) and \( (1/z_k)^* \) are also two conjugate zeros.

We gain simplifications by forming a fourth order section of the FIR system as follows

\[ H_1(z) = c_{k0} (1 - z_k z^{-1})(1 - z_k^* z^{-1})(1 - z^{-1} / z_k)(1 - z^{-1} / z_k^*) \]

\[ H_k(z) = c_{k0} + c_{k1}z^{-1} + c_{k2}z^{-2} + c_{k3}z^{-3} + c_{k4}z^{-4} \]
Cascade Form Structures

\[ x_k(n) \rightarrow z^{-1} \rightarrow z^{-1} \rightarrow \ldots \rightarrow y_k(n) \]

Frequency Sampling Structures

The frequency response of FIR filter is

\[ H(\omega) = \sum_{n=0}^{M-1} h(n)e^{-j\omega n} \]

If we defined the desired frequency response at set of equally spaced frequencies or at

\[ \omega_k = \frac{2\pi}{M} (k + \alpha) \quad k = 0, 1, 2, \ldots \ldots \ldots \ldots \ldots \ldots M - 1 \]

\[ \alpha = 0 \text{ or } \alpha = 1/2 \]
The frequency response of FIR filter is

\[ H(k + \alpha) = H(\omega) \bigg|_{\omega = \frac{2\pi k}{M}} = \sum_{n=0}^{M-1} h(n)e^{-j2\pi kn/M} \quad k = 0, 1, \ldots, M - 1 \]

Which is M-point DFT for \( h(n) \) by performing the IDFT

\[ h(n) = \frac{1}{M} \sum_{k=0}^{M-1} H(k)e^{j2\pi kn/M} \quad n = 0, 1, 2, \ldots, M - 1 \]

The z-transform of \( h(n) \) is

\[ H(z) = \sum_{n=0}^{M-1} h(n)z^{-n} = \frac{1}{M} \sum_{k=0}^{M-1} H(k)e^{j2\pi kn/M} \bigg|_{z^{-1}} \]

This can be viewed as cascade of two systems the first is

\[ H_1(z) = \frac{1 - (z^{-M})}{M} \]
Frequency Sampling Structures

And the 2nd system is

\[ H_2(z) = \sum_{k=0}^{M-1} \frac{H(k)}{1 - z^{-1} e^{-j\frac{2\pi k}{M}}} \quad k = 0, \ldots, M-1 \]

For Linear Phase

\[ H(k) = H^*(M-k) \]

As a result of the symmetry, a pair of single pole filters can be combined to form a single two-pole filter with

\[ H_3(z) = \frac{H(0)}{1 - z^{-1}} + \sum_{\ell=1}^{(M-1)/2} \frac{A(k) + B(k)z^{-1}}{1 - 2 \cos(2\pi k / M) z^{-1} + z^{-2}} \quad \text{M odd} \]

A model of a filter with nonlinear phase and designed by frequency sampling is given in the next slide. The model of linear-phase is given in the example later on.
Example

Example: Sketch the block diagram for the direct form realization and the frequency sampling realization of the $M = 32$, Linear-Phase FIR filter $M = 32$ and $\alpha = 0$

$$H(\frac{2\pi}{32}) = \begin{cases} 1, & k = 0,1,2 \\ 0.5, & k = 3 \\ 0 & k = 4,5,\ldots,15 \end{cases}$$

We can exploit symmetry to obtain the direct form.
Example

For a linear phase filter, the direct form is:

\[ x(n) \rightarrow z^{-1} \rightarrow z^{-1} \rightarrow z^{-1} \rightarrow z^{-1} \rightarrow \ldots \rightarrow z^{-1} \rightarrow y(n) \]

where the coefficients are \( h(0), h(1), h(2), h(3), \ldots, h(15) \).