Butterworth Filter

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Filter types

What are the function of Filters?

A filter is a system that allow certain frequency to pass to its output and reject all other signals

Filters can be classified according to range of signal frequencies in the passband

- Lowpass filter
- Highpass filter
- Bandpass filter
- Stopband (bandreject) filter
Filter types according to its frequency response

- Butterworth filter
- Chebychev I filter
- Chebychev II filter
- Elliptic filter
**Butterworth filter**

Ideal lowpass filter is shown in the figure

The passband is normalised to one.
Tolerance in passband and stopband are allowed to enable the construction of the filter.

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**Lowpass prototype filter**

Lowpass prototype filter: it is a lowpass filter with cutoff frequency $\Omega_p = 1$.

The frequency scale is normalized by $\omega_p$. We use $\Omega = \omega / \omega_p$. 
**Lowpass prototype filter**

**Notation**
In analogue filter design we will use

- $s$ to denote complex frequency
- $\omega$ to denote analogue frequency
- $p$ to denote complex frequency at lowpass prototype frequencies.
- $\Omega$ to denote analogue frequency at the lowpass prototype frequencies.

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**Magnitude Approximation of Analog Filters**

- The transfer function of analogue filter is given as rational function of the form
  \[ H(s) = \frac{c_0 + c_1 s + c_2 s^2 + \cdots + c_m s^m}{d_0 + d_1 s + d_2 s^2 + \cdots + d_n s^n} \quad m \leq n \]

- The Fourier transform is given by
  \[ H(\omega) = H(s) \Big|_{s=j\omega} = \frac{c_0 + j c_1 \omega - c_2 \omega^2 + \cdots + (j)^m c_m \omega^m}{d_0 + j d_1 \omega - d_2 \omega^2 + \cdots + (j)^n d_n \omega^n} \]
  \[ H(\omega) = |H(j\omega)| e^{j\phi(\omega)} \]
Magnitude Approximation of Analog Filters

- Analogue filter is usually expressed in term of
  \[ |H(j\omega)|^2 = H(j\omega)H^*(j\omega) \]
  \[ \angle H(j\omega) = -2\phi(\omega) \]

- Example
  Consider the transfer function of analogue filter, find
  \[ H(s) = \frac{s + 1}{s^2 + 2s + 2} \]
  \[ |H(j\omega)|^2 = H(s)H(-s) = \frac{s + 1}{s^2 + 2s + 2} \frac{-s + 1}{s^2 - 2s + 2} \]
  \[ |H(j\omega)|^2 = \frac{s^2 + 1}{s^2 - 2s + 4} \]

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\[ |H(j\omega)|^2 \] Will have only even powers of \( \omega \) or
\[ |H(\omega)|^2 = \frac{C_2 + C_4\omega^2 + C_6\omega^4 + \cdots + C_{2n}\omega^{2n}}{1 + D_2\omega^2 + D_4\omega^4 + \cdots + D_{2n}\omega^{2n}} \]

In order to approximate the ideal filter

1) The magnitude at \( \omega = 0 \) is normalized to one
2) The magnitude monotonically decreases from this value to zero as \( \omega \to \infty \).
3) The maximum number of its derivatives evaluated at \( \omega = 0 \) are zeros.
   This can be satisfied if
   \[ |H(\omega)|^2 = \frac{1}{1 + D_{2n}\omega^{2n}} \]
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The following specification is usually given for a lowpass Butterworth filter is:

1) The magnitude of $H_0$ at $\omega = 0$
2) The bandwidth $\omega_p$
3) The magnitude at the bandwidth $\omega_p$
4) The stopband frequency $\omega_s$
5) The magnitude at the stopband frequency $\omega_s$
6) The transfer function is given by

$$|H(\Omega)|^2 = \frac{H_0}{1 + D_{2N}(\Omega)^{2\epsilon}}$$

Butterworth filter

To achieve the equivalent lowpass prototype filter

1) We scale the cutoff frequency to one using transformation $\Omega = \omega / \omega_p$.
2) We scale the magnitude to 1 to 1 by dividing the magnitude by $H_0$.

The transfer function become

$$|H(\Omega)|^2 = \frac{1}{1 + D_{2N}(\Omega)^{2\epsilon}}$$

We denotes $D_{2N}$ as $\epsilon^2$ where $\epsilon$ is the ripple factor, then

$$|H(\Omega)|^2 = \frac{1}{1 + \epsilon^2(\Omega)^2}$$
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If the magnitude at the bandwidth $\Omega = \Omega_p = 1$ is given as $(1 - \delta_p)^2$ or $-A_p$ decibels, the value of $\varepsilon^2$ is computed by:

$$20 \log |H(\omega)|_{\omega = \Omega_p} = -2A_p$$
$$10 \log \frac{1}{1 + \varepsilon^2} = -A_p$$
$$\varepsilon^2 = 10^{A_p/20} - 1$$

If we choose $A_p = -3$ dB $\Rightarrow \varepsilon^2 = 1$, this is the most common case and gives

$$|H(\omega)|^2 = \frac{1}{1 + (\omega/\Omega)^2}$$

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If we use the complex frequency representation

$$|H(p)|^2 = |H(\omega)|^2 \bigg|_{\omega = \Omega} = \frac{1}{1 + (p/\Omega)^2}$$

The poles of this function occurs at

$$p_k = \begin{cases} e^{\pm \frac{2\pi j k}{2N}} & k = 1, 2, ..., 2N, \text{ n odd} \\ e^{\pm \frac{\pi j (2N-k)}{2N}} & k = 1, 2, ..., 2N, \text{ n even} \end{cases}$$

Or in general

$$p_k = e^{\pm \frac{2\pi j (N-1)k}{2N}} \quad k = 1, 2, ..., 2N$$

Poles occurs in complex conjugates

Poles which are located in the LHP are the poles of $H(s)$

$$p_k = e^{\pm \frac{2\pi j (N-1)k}{2N}} \quad k = 1, 2, ..., N$$
When we found the $N$ poles we can construct the filter transfer function as

$$H(p) = \frac{1}{D(p)}$$

The denominator polynomial $D(p)$ is calculated by

$$D(p) = \prod_{k=1}^{N} (p - p_k)$$

Another method to calculate $D(p)$ using

$$D(p) = \prod_{k=1}^{N} (p - p_k)$$

$$D(p) = 1 + d_1 p + d_2 p^2 + \cdots + d_N p^N$$

$$d_k = \frac{\cos\left(\frac{(k-1)\pi}{2}\right)}{\sin\left(\frac{k\pi}{2N}\right)} d_{k-1}, \quad k = 1, 2, 3, \ldots, N$$

The coefficients $d_k$ is calculated recursively where $d_0 = 1$.
**Butterworth filter**

The minimum attenuation as dB is usually given at certain frequency $\Omega_s$.

The order of the filter can be calculated from the filter equation:

$$10 \log |H(\Omega_s)|^2 = -A_s$$

$$-10 \log (1 + \Omega_s^{2N})$$

$$N \geq \log \left[ \frac{10^{-A_s/10} - 1}{2 \log(\Omega_s)} \right]$$

**Design Steps of Butterworth Filter**

1. Convert the filter specifications to their equivalents in the lowpass prototype frequency.
2. From $A_p$ determine the ripple factor $\varepsilon$.
3. From $A_s$ determine the filter order, $N$.
4. Determine the left-hand poles, using the equations given.
5. Construct the lowpass prototype filter transfer function.
6. Use the frequency transformation to convert the LP prototype filter to the given specifications.
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Example:
Design a lowpass Butterworth filter with a maximum gain of 5 dB and a cutoff frequency of 1000 rad/s at which the gain is at least 2 dB and a stopband frequency of 5000 rad/s at which the magnitude is required to be less than −25 dB.

Solution:

\[ \omega_p = 1000 \text{ rad/s} \quad \omega_s = 5000 \text{ rad/s} \]

By normalization,

\[ \Omega_p = \frac{\omega_p}{\omega_s} = 1 \text{ rad/s} \]
\[ \Omega_s = \frac{\omega_s}{\omega_p} = 5 \text{ rad/s} \]

And the stopband attenuation \( A_s = 25 + 5 = 30 \text{ dB} \)

The filter order is calculated by

\[ N = \frac{\log(10^{5/3}) - 1}{2\log(10)} = [2.146] = 3 \]

Hence the transfer function of the normalized prototype filter of third order is

\[ H(p) = \frac{1}{p^3 + 2p^2 + 2p + 1} \]
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To restore the magnitude, we multiply be $H_0$

$$H(p) = \frac{H_0}{p^3 + 2p^2 + p + 1}$$

$20\log H_0 = 5\text{dB}$ which leads $H_0 = 1.7783$

To restore the frequency we replace $p$ by $s/1000$

$$H(s) = H(p)\bigg|_{p=s/1000} = \frac{1.7783}{\left(\frac{s}{1000}\right)^3 + 2\left(\frac{s}{1000}\right)^2 + 2\left(\frac{s}{1000}\right) + 1}$$

$$H(s) = \frac{1.7783 \times 10^9}{s^3 + 2000s^2 + 2 \times 10^7 s + 10^5}$$

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If the passband edge is defined for $A_p \neq 3\text{ dB}$ (i.e. $\varepsilon \neq 1$).

The design equation needs to be modified. The formula for calculating the order will become

$$N \geq \frac{\log\left[10^{4.10^{-1}} - 1\right]\varepsilon}{2\log(\Omega_p)}$$

And the poles are given by

$$p_k = e^{-jN/k} e^{j(2N+3\pi)/2N} \quad k = 1,2,\ldots,N$$

Home Study: Repeat the previous example if $A_p = 0.5\text{ dB}$