Chebyshev Filters

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Chebyshev I filter

It has ripples in the passband and no ripple in the stop band
It has the following transfer function

$$|H(\Omega)|^2 = \frac{1}{1 + \varepsilon^2 C_n(\Omega)}$$

$C_n(x)$ is $N^{th}$ order Chebyshev polynomial defined as

$$C_n(x) = \begin{cases} \cos(N \cos^{-1} x) & |x| \leq 1 \\ \cosh(N \cosh^{-1} x) & |x| > 1 \end{cases}$$

The Chebyshev polynomial can be generated by recursive technique

$$C_0(\Omega) = 1 \quad \text{and} \quad C_1(\Omega) = \Omega$$
$$C_{n+1}(\Omega) = 2\Omega C_n(\Omega) - C_{n-1}(\Omega)$$
Chebyshev polynomial

Chebyshev polynomial characteristics

\[ C_n(\Omega) \leq 1 \text{ for all } |\Omega| \leq 1 \]

\[ C_n(1) = 1 \text{ for all } N \]

All the roots of \( C_n(\Omega) \) occurs in the interval \(-1 \leq x \leq 1\)

The filter parameter \( \varepsilon \) is related to the ripple as shown in the figures

\[ H(\varepsilon) = \begin{cases} 
\frac{1}{1 + \varepsilon^2} & \text{N odd} \\
\frac{1}{1 + \varepsilon^2} & \text{N even}
\end{cases} \]

Filter roots

The poles of LP prototype filter \( H(p) \) are given by

\[ p_k = -\sinh(\varphi_2)\sin(\theta_k) + j\cosh(\varphi_2)\cos(\theta_k) \quad k = 1,2,3,\ldots,N \]

\[ \varphi_2 = \frac{1}{N} \sinh^{-1}\left(\frac{1}{\varepsilon}\right) \]

\[ \theta_k = \frac{(2k-1)\pi}{2N} \]

And the order of the filter is given by

\[ N \geq \frac{\cosh^{-1}\sqrt{\left|\frac{10^{\varepsilon/10} - 1}{10^{\varepsilon/10} - 1}\right|}}{\cosh^{-1}\Omega_c} \]
Chebyshev filter type I

Example: Design lowpass Chebyshev filter with maximum gain of 5 dB, \( A_p = 0.5 \) dB ripple in the passband, a bandwidth of 2500 rad/sec. A stopband frequency of 12.5 Krad/sec, and \( A_s \geq 30 \) dB or more for \( \Omega > \Omega_s \)

\[ \omega_p = 2500 \text{ rad/s}, \quad \omega_s = 12.5 \text{ Krad/s}, \]

For the LP prototype filter \( A_s = 30 \) dB, \( A_p = 0.5 \) dB

\[ \Omega_p = \frac{\omega_p}{\omega_p} = 1 \text{ rad/s} \]
\[ \Omega_s = \frac{\omega_s}{\omega_p} = 5 \text{ rad/s} \]

To determine the order of the filter

\[ N = \frac{\cosh^{-1}\sqrt{(10^5 - 1)/(10^{10} - 1)}}{\cosh^{-1}(5)} = \lceil 2.2676 \rceil = 3 \]

To calculate the ripple factor

\[ 10 \log(1 + \epsilon^2) = 0.5 \]
\[ \epsilon = 0.34931 \]

Chebyshev filter type I

\[ \varphi_1 = \frac{1}{2} \sinh^{-1}\left( \frac{1}{0.34931} \right) = 0.591378 \]

\[ p_s = -0.313228 \pm j1.02192 \quad \text{and} \quad -0.626456 \]

The transfer function

\[ H(p) = \frac{H_0}{(p^2 + 0.626456p + 1.142447)(p + 0.62645)} \]

To determine \( H_0 \) we know that \( H(p = 0) = 1 \) (\( N \) is odd)

\[ 1 = \frac{H_0}{(1.142447)(0.62645)} \quad H_0 = 0.715693 \]

And to have 5 dB gain we multiply \( H_0 \) by 1.7783

\[ H(p) = \frac{(0.715693)(1.7783)}{(p^2 + 0.626456p + 1.142447)(p + 0.62645)} \]
Chebyshev filter type I

When we substitute by $p = s/2500$

$$H(s) = \frac{19.886 \times 10^{12}}{s^2 + 1566s + 714 \times 10^6(s + 1566)}$$

Chebyshev filter type II

This class of filters is called inverse Chebyshev filters. Its transfer function is derived by applying the following:

1) Frequency transformation of $\Omega' = 1/\Omega$ in $H(j\Omega)$ of lowpass normalized prototype filter to achieve the following highpass filter

$$\left| H\left(\frac{1}{\Omega}\right) \right|^2 = \frac{1}{1 + e^{-2C_n}(1/\Omega')}$$

2) When it subtracted from one we get transfer function of Chebyshev II filter

$$1 - \frac{1}{1 + e^{-2C_n}(1/\Omega')} = \frac{1}{1 + e^{-2C_n}(1/\Omega')}$$
Chebyshev filter type II

It has ripples in the passband and ripple in the stop band
It has the following transfer function

Chebyshev filter type II Design

1. The frequencies are normalised by $\omega_s$ instead of $\omega_p$.
2. The ripple factor $\varepsilon_i$ is calculated by

\[ \varepsilon_i = 1 / \sqrt{\left(10^{0.1k} - 1\right)} \]

2. The order of the filter is given by

\[ N \geq \text{cosh}^{-1}\left(\frac{\text{cosh}^{-1}(1/\Omega_p)}{\text{cosh}^{-1}(1/\Omega_p)}\right) \]

3. The poles of LP prototype filter $H(p)$ are given by

\[ p_k = 1 / \left[\sinh(\phi_k)\sin(\theta_k) + j \cosh(\phi_k)\cos(\theta_k)\right] \quad k = 1, 2, 3, \ldots, N \]

\[ \phi_k = \frac{1}{N} \sinh^{-1}\left(\frac{1}{\varepsilon}\right) \quad \theta_k = \frac{(2k-1)\pi}{2N} \]
Chebyshev filter type II Design

5. The zeros of $H(p)$ is given by
   
   $$z_k = \pm \Omega_{m} = \pm j \sec \theta_k \quad k = 1, 2, 3, \ldots, m = \left\lfloor N/2 \right\rfloor$$

   The filter transfer function is calculated by
   
   $$H(p) = H_0 \prod_{k=1}^{m} \left( p^2 + \Omega_{m}^2 \right) / \prod_{k=1}^{m} \left( p - p_k \right)$$

6. Restore the magnitude scale by calculating $H_0$.
7. Restore the frequency scale to $\omega$ by frequency transformation

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Chebyshev filter type II

Example: Design lowpass Chebyshev II filter that has 0-dB ripple in the passband, $\omega_p = 1000 \text{ rad/sec}$, $A_p = 0.5 \text{ dB}$, a stopband frequency of 2000 rad/sec and $A_s = 40 \text{ dB}$.

$\omega_p = 1000 \text{ rad/s}, \quad \omega_s = 2000 \text{ krad/s}$,

For the LP prototype filter

$\Omega_p = \omega_p / \omega_s = 0.5 \text{ rad/s}$

$\Omega_s = \omega_s / \omega_s = 1 \text{ rad/s}$

$10 \log(1 + \epsilon^2) = 40$

$\epsilon = 1/99.995$

To determine the order of the filter

$$N = \frac{\cosh^{-1} \sqrt{\left\lfloor 10^{4} - 1 \right\rfloor / \left\lfloor 10^{0.03} - 1 \right\rfloor}}{\cosh^{-1}(2)} = \left\lfloor 4.82 \right\rfloor = 5$$
Chebyshev filter type II

\[ \varphi_2 = \frac{1}{5} \sinh^{-1}(99.995) = 1.05965847. \]

\[ p_k = (-0.155955926 \pm j0.6108703175), (-0.524799485 \pm j0.485389011), \]
and \((-0.7877702666)\)

The zeros is calculated by

\[ z_k = \pm j\Omega_{\text{z}_k} = j \sec \Theta_k \quad k = 1, 2, \ldots, m = \lfloor n/2 \rfloor \]

\[ z_1 = \pm j1.0515 \quad z_2 = \pm j1.7013 \]

The transfer function

\[ H(p) = \frac{H_0 \left(p^2 + 1.0515^2\right) \left(p^2 + 1.7013^2\right)}{\left(p^2 + 0.3118311852 p + 0.3974722176\right) \left(p^2 + 0.04959897 p + 0.5110169847\right) \left(p + 0.787702666\right)} \]

Chebyshev filter type II

\[ H(p) = \frac{H_0 \left(p^4 + 4.04 p^2 + 3.2002\right)}{p^5 + 2.1491328 p^4 + 2.30818905 p^3 + 154997 p^2 + 0.657255515 p + 0.15999426} \]

To determine \( H_0 \) we know that \( H(p=0) = 1 \) (\( N \) is odd)

\[ 1 = \frac{3.2002 H_0}{0.15999426} \]

\[ H_0 = 0.049995 \]

\( H(s) \) can be achieved by substituting by \( p = s/2000 \)
Elliptic filter

Elliptic filter is more efficient than Butterworth and Chebychev filters as it has the smallest order for a given set of specifications.

The drawback it suffers from high nonlinearity in the phase, especially near the band edges.

The filter order requires to achieve a given set of specifications in passband ripple $\delta_1$ and $\delta_2$. 