Discrete Mathematics

Chapter 8

Relations
8.1: Relations and there prosperities:

- Definitions:
  1. Let A and B be sets. A binary relation from A to B is a subset of A x B.
  2. A relation on the set A is a relation from A to A.
  3. A relation R on a set A is called reflexive if (a, a) ∈ R for every element a ∈ A.
  4. A relation R on the set A is irreflexive if for every a ∈ A, (a, a) /∈ R.
  5. A relation R on a set A is called symmetric if (b, a) ∈ R whenever (a, b) ∈ R, for all a, b ∈ A. A relation R on a set A such that for all a, b ∈ A, if (a, b) ∈ R and (b, a) ∈ R, then a = b is called antisymmetric.
  6. A relation R on a set A is called transitive if whenever (a, b) ∈ R and (b, c) ∈ R, then (a, c) ∈ R, for all a, b, c ∈ A.
  7. Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where a ∈ A, c ∈ C, and for which there exists an element b ∈ B such that (a, b) ∈ R and (b, c) ∈ S. We denote the composite of R and S by S o R.
Exercises

1. 8.1.3 For each of these relations on the set {1, 2, 3, 4}, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

   a.  \{ (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4) \}
   b.  \{ (1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4) \}
   c.  \{ (2, 4), (4, 2) \}
   d.  \{ (1, 2), (2, 3), (3, 4) \}
   e.  \{ (1, 1), (2, 2), (3, 3), (4, 4) \}
   f.  \{ (1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4) \}

   Solution:

<table>
<thead>
<tr>
<th></th>
<th>reflexive</th>
<th>irreflexive</th>
<th>symmetric</th>
<th>antisymmetric</th>
<th>asymmetric</th>
<th>transitive</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>false</td>
<td>false</td>
<td>false</td>
<td>false</td>
<td>false</td>
<td>true</td>
</tr>
<tr>
<td>b</td>
<td>true</td>
<td>false</td>
<td>true</td>
<td>false</td>
<td>false</td>
<td>true</td>
</tr>
<tr>
<td>c</td>
<td>false</td>
<td>true</td>
<td>true</td>
<td>false</td>
<td>false</td>
<td>false</td>
</tr>
<tr>
<td>d</td>
<td>false</td>
<td>true</td>
<td>false</td>
<td>true</td>
<td>true</td>
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</tr>
<tr>
<td>e</td>
<td>true</td>
<td>false</td>
<td>true</td>
<td>true</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>f</td>
<td>false</td>
<td>true</td>
<td>false</td>
<td>false</td>
<td>false</td>
<td>false</td>
</tr>
</tbody>
</table>

2. 8.1.30 Let R be the relation \{ (1, 2), (1, 3), (2, 3), (2, 4), (3, 1) \}, and let S be the relation \{ (2, 1), (3, 1), (3, 2), (4, 2) \}. Find S o R.

   Solution:

   \[
   \begin{array}{ccc}
   1 & 2 & 1 \\
   1 & 3 & 1 \\
   1 & 3 & 2 \\
   2 & 3 & 1 \\
   2 & 3 & 2 \\
   2 & 4 & 2 \\
   3 & 1 & \\
   \end{array}
   \]

   The result contain S o R = \{ (1, 1), (1, 2), (2, 1), (2, 2) \}
8.3: Representing Relations:

- The relation \( R \) can be represented by the matrix \( M_R = [m_{ij}] \), where:
  \[
  m_{ij} = \begin{cases} 
  1 & \text{if } (a_i, b_j) \in R, \\
  0 & \text{if } (a_i, b_j) \notin R.
  \end{cases}
  \]

- A directed graph, or digraph, consists of a set \( V \) of vertices (or nodes) together with a set \( E \) of ordered pairs of elements of \( V \) called edges (or arcs). The vertex \( a \) is called the initial vertex of the edge \((a, b)\), and the vertex \( b \) is called the terminal vertex of this edge.

![Diagrams showing reflexive, irreflexive, symmetric, and antisymmetric properties of relations.](image_url)
Exercises

1. 8.3.4 List the ordered pairs in the relations on \{1, 2, 3, 4\} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

\[
a) \begin{bmatrix}
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
\end{bmatrix} \\
b) \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Solution:

a) \{ (1,1), (1,2), (1,4), (2,1), (2,3), (3,2), (3,3), (3,4), (4,1), (4,3), (4,4) \}

b) \{ (1,1), (1,2), (1,3), (2,2), (3,3), (3,4), (4,1), (4,4) \}

<table>
<thead>
<tr>
<th></th>
<th>reflexive</th>
<th>irreflexive</th>
<th>symmetric</th>
<th>antisymmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>false</td>
<td>false</td>
<td>true</td>
<td>false</td>
</tr>
<tr>
<td>b</td>
<td>true</td>
<td>false</td>
<td>false</td>
<td>true</td>
</tr>
</tbody>
</table>

2. 8.3.23-26 List the ordered pairs in the relations represented by the directed graphs.

Solution:

1. \{ (a, c), (a, b), (b, c), (c, b) \}

2. \{ (a, a), (a, b), (b, b), (b, a), (c, c), (c, a), (c, d), (d, d) \}

<table>
<thead>
<tr>
<th></th>
<th>reflexive</th>
<th>irreflexive</th>
<th>symmetric</th>
<th>antisymmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>false</td>
<td>true</td>
<td>false</td>
<td>false</td>
</tr>
<tr>
<td>b</td>
<td>true</td>
<td>false</td>
<td>false</td>
<td>false</td>
</tr>
</tbody>
</table>
8.4: Closures of Relations

- For any property $X$, the “$X$ closure” of a set $A$ is defined as the “smallest” superset of $A$ that has the given property.
- The reflexive closure of a relation $R$ on $A$ is obtained by adding $(a, a)$ to $R$ for each $a \in A$. I.e., it is $R \cup I_A$.
- The symmetric closure of $R$ is obtained by adding $(b, a)$ to $R$ for each $(a, b)$ in $R$. I.e., it is $R \cup R^{-1}$.
- The transitive closure or connectivity relation of $R$ is obtained by repeatedly adding $(a, c)$ to $R$ for each $(a, b), (b, c)$ in $R$.

- A path from $a$ to $b$ in the directed graph $G$ is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in $G$, where $n$ is a nonnegative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has length $n$. We view the empty set of edges as a path from $a$ to $a$. A path of length $n \geq 1$ that begins and ends at the same vertex is called a circuit or cycle.

- A path of length $n \geq 1$ from $a$ to $a$ is called a circuit or a cycle.
- There exists a path of length $n$ from $a$ to $b$ in $R$ if and only if $(a, b) \in R^n$. 
Exercises

1. Let $R$ be the relation on the set $\{0, 1, 2, 3\}$ containing the ordered pairs $(0, 1), (1, 1), (1, 2), (2, 0), (2, 2),$ and $(3, 0)$. Find the

   a. Reflexive closure of $R$.
   b. Symmetric closure of $R$.

Solution:

a. Reflexive closure of $R = R \cup I_A = \{(0, 0), (0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

b. Symmetric closure of $R = R \cup R^{-1} = \{(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)\}$
8.5: Equivalence Relations:

- An equivalence relation (e.r.) on a set $A$ is simply any binary relation on $A$ that is reflexive, symmetric, and transitive.
- Let $R$ be an equivalence relation on a set $A$. The set of all elements that are related to an element $a$ of $A$ is called the equivalence class of $a$. The equivalence class of $a$ with respect to $R$ is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript $R$ and write $[a]$ for this equivalence class.
- A partition of a set $A$ is the set of all the equivalence classes $\{A_1, A_2, \ldots \}$ for some e.r. on $A$.

Exercises

1. 8.5.1 Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

   a. $\{(0, 0), (1,1), (2, 2), (3, 3)\}$
   b. $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$
   c. $\{(0, 0), (1,1), (1, 2), (2,1), (2, 2), (3, 3)\}$
   d. $\{(0, 0), (1,1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
   e. $\{(0, 0), (0,1), (0, 2), (1, 0), (1,1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

Solution:

<table>
<thead>
<tr>
<th>Equivalence Relation</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>a  True</td>
<td></td>
</tr>
<tr>
<td>b  False</td>
<td>not reflexive, (1,1) not exist</td>
</tr>
<tr>
<td>c  True</td>
<td></td>
</tr>
<tr>
<td>d  False</td>
<td>Not transitive, (1,3) and (3,2) but no (1,2)</td>
</tr>
<tr>
<td>e  False</td>
<td>Not symmetric, (1,2) but no (2,1)</td>
</tr>
</tbody>
</table>