Chapter 1
Solving Nonlinear Equations

In this chapter we study methods to solve nonlinear equations of the form

\[ f(x) = 0. \]

A solution of \( f(x) = 0 \) is also called a root or a zero of \( f \).

1.2. Bisection Method (Interval Halving)

The bisection method is an application of the intermediate value theorem.

**Theorem 1. (intermediate value theorem)**

Let \( f(x) \) be a continuous function on an closed interval \([a, b]\) and let \( y_0 \) be a number between \( f(a) \) and \( f(b) \). Then there is a number \( c \in (a, b) \) such that \( f(c) = y_0 \).

Our aim is to solve \( f(x) = 0 \). That is to find a number \( c \) such that \( f(c) = 0 \). If we apply Intermediate value theorem with \( y_0 = 0 \), then we must have \( c \in (a, b) \) such that \( f(c) = y_0 = 0 \).

**Corollary 1.** If \( f(x) \) is continuous on \([a, b]\) with \( f(a)f(b) < 0 \), then there is a number \( c \in (a, b) \) such that \( f(c) = 0 \).

**Description of bisection method**

1. Find an interval \([a, b]\) such that \( f(a)f(b) < 0 \). This can be done by sketching the graph of \( f \) roughly or by try and error.

2. Define \( c = \frac{a + b}{2} \). Then \( c \) will be an approximation for the solution of \( f(x) = 0 \).

3. To find a better approximation we take a new interval as follows:
   - if \( f(c)f(a) < 0 \), then \( b = c \);
   - if \( f(c)f(a) > 0 \), then \( a = c \).

4. Repeat steps (2) and (3) until the approximation \( c \) is as good as we require.

**How to stop the method**

We can control the accuracy of the solution by one of the following two methods:

- We may stop when \( |a - b| < 2h \), where \( h \) is the allowed error.
- We may stop when \( |f(c)| < h \) for some small number \( h \). (\( f(c) \approx 0 \)).
Example 1. Use the bisection method to find a root of $f(x) = x^2 - 3$ in the interval $[1, 2]$. Stop when $|b - a| < 0.2$.

Solution:
Example 2. Use the bisection method to find a root of \( f(x) = e^x - 4x^2 \) in the interval \([0, 1]\). Stop when \(|f(x)| < 0.01\).

Solution:
**Error analysis**

Bisection method is one of the few methods of numerical methods where we know in advance how many steps we must take to reduce the error to a prescribed size.

In order to have $E_a < h$, we have to choose the integer $n$ such that

$$n > \frac{1}{\ln 2} \ln \left( \frac{b - a}{h} \right).$$

**Proof:**

**Example 1.** How many steps (iterations) do we need in bisection method to find an approximation for the root of $f(x) = e^x - 4x^2$ with absolute error smaller than 0.01 if we start with the interval $[0, 1]$.

**Solution:**

**Remark.** The bisection method converges very slowly.