1.10 Multiple Roots

Definition. We say that \( r \) is a root of multiplicity \( k \) of a function \( f(x) \) if
\[
f(r) = f'(r) = \cdots = f^{(k-1)}(r) = 0 \quad \text{and} \quad f^{(k)}(r) \neq 0.
\]

Example 1. \( r = 0 \) is a root of multiplicity 2 of \( f(x) = x^2 \).

Example 2. \( r = 1 \) is a root of multiplicity 4 of \( f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1 \).

Example 3. \( r = 0 \) is a root of multiplicity 2 of \( f(x) = x \sin x \).

Definition. (Multiple and simple roots)
Let \( r \) be a root of multiplicity \( k \) of a function \( f \).

(1) If \( k = 1 \), then \( r \) is called a simple root.

(2) If \( k > 1 \), then \( r \) is called a multiple root.

How can we find a multiple root?

Some of the methods we have studied always work with multiple roots, and some of them may or may not work. The bisection and false position methods do not work to get multiple roots because the function may not change sign at a multiple root. Secant, Newton’s, and fixed-point methods work to get multiple roots, but the convergence is slower than that for a simple root.

Example 1. Use Newton’s method to find the double root of \( f(x) = xe^x - x \). Start with \( x_0 = 0.2 \) and stop after 3 iterations.

Solution:
Remark. In addition to the slow convergence, another disadvantage of using Newton’s or secant method to find multiple roots is that \( f'(x) \) (or its approximation) will be near zero. Thus almost all computers will find \( f'(x) = 0 \) because of the imprecise arithmetic, and hence we will have division by zero.

**Modification of Newton’s Method**

One way to modify Newton’s is to find the root of \( f'(x) \) instead of \( f(x) \). Thus to find a double root we may use the formula

\[
x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}.
\]

**Example 1.** Use the modified Newton’s method \( x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} \) to find the double root of \( f(x) = x e^x - x \). Start with \( x_0 = 0.2 \) and stop after 3 iterations.

**Solution:**

**The Modified Newton’s Method**

Another way to modify Newton’s method is to use the formula

\[
x_{n+1} = x_n - k \frac{f(x_n)}{f'(x_n)},
\]

where \( k \) is the multiplicity of the root.
Remark.

(1) In the first modification we have the difficulty of calculating $f''(x)$ or other higher-order derivatives.

(2) In the second modification, we have the difficulty of knowing the multiplicity $k$ of the root. One way to overcome this difficulty is to try several values for $k$ until you have fast convergence.

Example 1. Use the modified Newton’s method $x_{n+1} = x_n - k \frac{f(x_n)}{f'(x_n)}$ to find the double root of $f(x) = xe^x - x$. Start with $x_0 = 0.2$ and compare the result after 3 iterations with Newton’s method.

Solution:

Nearly Multiple Roots

A problem that is related to multiple roots is a function that has two roots or more roots very close to each other. In this case also Newton’s method converges slower if we start outside the interval containing the roots. If we start with a value between two almost equal roots, Newton’s method may diverge.

Remark. The modifications of Newton’s method do not work here; often an infinite loop occurs.