CHAPTER 2

LINEAR MOTION
Motion of an object is the continuous change in the position of that object. In this chapter we shall consider the motion of a particle in a straight line, which will be taken to be one of the coordinate axes.

### 2.1 AVERAGE VELOCITY

Consider a particle moving along a straight line (x-axis) as in Figure 2.1(a). It starts at point P at \( t_i \) and finishes at point Q at a later time \( t_f \). The position-time graph for the same particle is shown in Figure 2.1(b). The average velocity is the slope of the straight line connecting the initial and the final points.

![Figure 2.1(a)](image)

**Figure 2.1** (a) A particle moves along the x-axis. It starts at point P at \( t_i \) and finishes at point Q at a later time \( t_f \). (b) The position-time graph for the same particle. The average velocity is the slope of the straight line connecting the initial and the final points.

From this relation, it is clear that the velocity has a dimension of length divided by time (L/T) with a unit of m/s, cm/s, and ft/s according to the SI, Gaussian, and British system, respectively. The average velocity, \( \bar{v} \), of the particle during the time interval \( \Delta t \) is now defined as the ratio of \( \Delta x \) to \( \Delta t \), or

\[
\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} \quad (2.1)
\]
particle starts from a point and return back to the same point, its displacement, and so its average velocity is zero. Fig 2.1(b) shows the variation of $x$ with $t$ where the average velocity is given by the slope of the straight line between points P and Q.

**Remark:** There is a difference between distance and displacement. **Distance**, a scalar quantity, is the actual long of the path traveled by a particle, but **displacement**, a vector quantity, is the shortest distance between the initial and the final positions of the particle.

**Speed** is the magnitude of the velocity. This means that the speed can never be negative. **The average speed** differs from average velocity in that it covers the total distance rather than the total displacement, that is

\[
\text{average speed} = \frac{\text{total distance}}{\Delta t} \tag{2.2}
\]

Although the velocity and the displacement are vectors, we do not need to write them as vectors since all vectors of this chapter have only one component.

### 2.2 INSTANTANEOUS VELOCITY

The instantaneous velocity, $v$, is defined as the value of $\bar{v}$ when $\Delta t$ approaches zero (becomes instant), that is

\[
v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \tag{2.3}
\]

In the position-time graph (Figure 2.2), $v$ at some instant is the slope of the tangent at that instant.
Example 2.1 The position of a particle varies with time according to \( x = t^2 + 3t \) with \( x \) in m and \( t \) in s.

a) Find \( \bar{v} \) for the interval \( t=0 \) to \( t=2 \) s

b) Find \( v \) at \( t=1.5 \) s

Solution a) to find \( x_i \) and \( x_f \), we have to substitute for \( t_i \) and \( t_f \) in the \( x\)-\( t \) relation, that is,

\[
x_i = (t_i)^2 + 3(t_i) = 0,
\]

and

\[
x_f = (t_f)^2 + 3(t_f) = 4 + 6 = 10 \text{ m}.
\]

Now, from Equation 2.1, we have

\[
\bar{v} = \frac{x_f - x_i}{t_f - t_i} = \frac{10 - 0}{2 - 0} = 5 \text{ m/s}.
\]

b) From Equation 2.3, we get

\[
v = \frac{dx}{dt} = 2t + 3.
\]
The instantaneous velocity at \( t = 1.5 \text{ s} \) is obtained by substituting for \( t = 1.5 \text{ s} \) in the last equation

\[
v = 2(1.5) + 3 = 6 \text{ m/s}
\]

### 2.3 ACCELERATION

When the velocity of a moving body changes with time, we say that the body has acceleration. The **average acceleration** \( \bar{a} \), during a time interval \( \Delta t \), is the ratio of the change in velocity, \( \Delta v \), to the time interval \( \Delta t \), i.e.,

\[
\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}
\]

(2.4)

The unit of the acceleration is \( \text{m/s}^2 \) in SI system.

In analog with the instantaneous velocity, the **instantaneous acceleration**, \( a \), is defined as the limit of the average acceleration as \( \Delta t \) approaches zero, i.e.

\[
a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}
\]

(2.5)

From Equation 2.5, it is clear that the acceleration is the slope of the velocity-time graph. As \( v = \frac{dx}{dt} \) (from Equation 2.3), then Equation 2.5 can be rewritten as

\[
a = \frac{dv}{dt} = \frac{d}{dt} \left( \frac{dx}{dt} \right) = \frac{d^2x}{dt^2}
\]

(2.6)
Example 2.2 The velocity of an object moving along the x-axis varies with time according to the relation $v = 5t - 3$ with $v$ in m/s and $t$ in s.

a) Find $\ddot{a}$ during the interval $t=1$ s to $t=2$ s

b) Find $a$ at $t=2$s

Solution

a) $v_i = 5(t_1) - 3 = 5(1) - 3 = 2 \text{ m/s}$,

and

$v_f = 5(t_f) - 3 = 5(2) - 3 = 7 \text{ m/s}$.

So, from Equation 2.4

$$\ddot{a} = \frac{v_f - v_i}{t_f - t_i} = \frac{7 - 2}{2 - 1} = 5 \text{ m/s}^2.$$

b) Using Equation 2.5, we obtain

$$a = \frac{dv}{dt} = 5 \text{ m/s}^2.$$

Example 2.3 The position-time graph of a particle moving along the x-axis is given in Figure 2.3. Find

a) $\ddot{x}$ during the interval $t=2$ s to $t=5$ s.

b) $\ddot{x}$ during the interval $t=0.5$ s to $t=2.5$ s

Solution a) As it is clear from the graph, at $t = 2$ s $x_i = 3 \text{ m}$, and at $t = 5$ s $x_f$
\( x_i = 0. \) Now from Equation 2.1, we get

\[
\bar{v} = \frac{x_f - x_i}{t_f - t_i} = \frac{0 - 3}{5 - 2} = -1 \text{ m/s}
\]

b) Since \( v \) at any point is the slope of the \( x-t \) graph at that point, we have at \( t=0.5 \text{ s} \) \( v=6 \text{ m/s} \), and at \( t=2.5 \text{ s} \) \( v=0 \), so

\[
\bar{a} = \frac{v_f - v_i}{t_f - t_i} = \frac{0 - 6}{2.5 - 0.5} = -3 \text{ m/s}^2
\]

\section*{v. LINEAR MOTION WITH CONSTANT ACCELERATION}

The simplest type of linear motion is the uniform motion in which the acceleration is constant. In such a case \( \bar{a} = a \), so from Equation 2.4, we obtain

\[
a = \frac{v_f - v_i}{t_f - t_i} = \frac{v - v_0}{t} \tag{2.7}
\]

where we denote \( v_i \) by \( v_0 \), \( v_f \) by \( v \), and \( t_f \) by \( t \), and take \( t_i=0 \). The above equation now is

\[
v = v_0 + at \tag{2.8}
\]

Also, because \( a \) is constant, we can write

\[
\bar{v} = \frac{v + v_0}{2} \tag{2.9}
\]

Using Equations 2.1 and 2.9, we obtain
\[ \frac{x - x_0}{t} = \frac{v + v_0}{2} \]  

(2.10)

Substituting for \( v \) from Equation 2.8 and rearrange, we obtain

\[ x - x_0 = v_0 t + \frac{1}{2} at^2 \]  

(2.11)

Now substituting for \( t \) from Equation 2.8 into Equation 2.10 we get

\[ v^2 = v_0^2 + 2a(x - x_0) \]  

(2.12)

When the velocity is constant, \((a=0)\) it is clear from Equations 2.8 and 2.11 that

\[ v = v_0 \text{ and } x - x_0 = vt \]  

(2.13)

For simplicity, the origin of the coordinates is often chosen to be coincident with \( x_0 \) (i.e. \( x_0=0 \)). Equations 2.8, 2.11, and 2.12 are the fundamental three equations that govern the linear motion with constant acceleration.

**Strategy for solving problems with constant acceleration:**

(i) Choose your coordinates such that the particle begins its motion from the origin \((x_0 =0)\).

(ii) Decide the sense of the positive direction.

(iii) Make a list of the known quantities. Do not forget to write any vector quantity \((x, v, v_0, a)\) that have a direction opposite to your positive sense as a negative quantity.

(iv) Make sure that all the quantities have the same system of units.
(v) According to what is given and what is requested, you can easily decide which equation or equations from equations 2.8, 2.11, and 2.12 you need to solve for the unknowns.

Example 2.4 A car starts from rest and moves with constant acceleration. After 12 s its velocity becomes 120 m/s. Find,
a) the acceleration of the car
b) the distance the car travels in the 12 s

Solution Let the direction of motion be along the positive $x$-axis, where the car starts from the origin at $t=0$. Now $v_0=0$, $v=120$ m/s, $t=12$ s.

a) Using Equation 2.8, namely $v = v_0 + at$, we have

$$a = \frac{v - v_0}{t} = \frac{120 - 0}{12} = 10 \text{ m/s}^2.$$

b) Equation 2.11 reads $x = v_0 t + \frac{1}{2} at^2$, substituting for $v_0$, $t$, and $a$ yields

$$x = 0 + (0.5)(10)(12)^2 = 720 \text{ m}.$$

2.5 FREELY FALLING BODIES

A freely falling body is any body moving freely under the influence of gravity regardless of its initial motion. Neglecting the air resistance and assuming that the gravitational acceleration, denoted by $g$, is constant, we can consider the motion of a freely body as a linear motion with constant acceleration. Taking your axis to be the $y$-axis with the positive sense upward, Equations 2.8, 2.11, and 2.12 will apply with the substitutions, $x \rightarrow y$ and $a \rightarrow -g$, i.e.,
\[ v = v_0 - gt \]  
(2.14)

\[ y - y_0 = v_0 t - \frac{1}{2} gt^2 \]  
(2.15)

\[ v^2 = v_0^2 - 2g(y - y_0) \]  
(2.16)

The first substitution is because the motion is now vertical and the negative sign in the second substitution indicates that the acceleration is downward.

**Remark:** Remember that the gravitational acceleration, \( g \), is constant in magnitude and in direction and this means that the negative sign of \( g \) in Equations 2.14-2.16 will not be changed unless you change your positive sense, regardless of the direction of motion.

**Example 2.5** An object is thrown vertically upward with an initial speed of 25 m/s.

a) How long does it take to reach its maximum height?

b) What is the maximum height?

c) How long does it take to return to the ground?

d) What is its velocity just before striking the ground?

**Solution** The track of the object is shown in Figure 2.5.

a) At the maximum point \( v = 0 \). From Equation 2.14 we have
\[ t = \frac{v_o - v}{g} = \frac{25 - 0}{9.8} = 2.55\text{s}. \]

b) Using Equation 2.16, i.e. \( v^2 = v_o^2 - 2gy \) we have

\[ h = y = \frac{v_o^2 - v^2}{2g} = \frac{(25)^2 - 0}{2(9.8)} = 31.9\text{m} \]

c) When returning to the ground, the displacement of the object is zero \((y=0)\). So from Equation 2.15, we have

\[ 0 = v_o t - \frac{1}{2} gt^2. \]

Solving for \( t \) we get

\[ t = \frac{2v_o}{g} = \frac{25}{4.9} = 5.1\text{s}. \]

d) From Equation (2.14) we have

\[ v = v_0 - gt = 25 - 9.8(5.1) = -25\text{m/s} \]

The minus sign indicates that \( v \) is directed downward.

**Example 2.6** A student, stands at the edge of the roof of a building, throws a ball vertically upward with an initial speed of 20 m/s. The building is 50 m high, and the ball just missed the edge of the roof in its way down, (Figure 2.6). Find,

a) the time needed for the ball to return to the level of the roof
b) the velocity and the position of the ball at \( t = 5 \text{ s} \)
c) the velocity of the ball just before hitting the ground

**Solution**

a) When the ball returns to the level of the roof, its displacement, \( y \), is zero. Substituting in Equation 2.15 we get

\[
0 = 20t - \frac{1}{2}(9.8)t^2.
\]

Now solving for \( t \) we get

\[
t = \frac{40}{9.8} = 4.08 \text{ s}.
\]

b) Using Equation 2.14, \( v = v_0 - gt \), we obtain

\[
v = 20 - 9.8(5) = -29 \text{ m/s},
\]

and using Equation 2.15, \( y = v_0 t - \frac{1}{2} gt^2 \), we have

\[
y = 20(5) - 4.9(25) = -22.5 \text{ m}.
\]

c) From the equation \( v^2 = v_0^2 - 2gy \), (Equation 2.16), we obtain

\[
v^2 = (20)^2 - 2(9.8)(-50)
= 400 + 980 = 1380 \text{ m}^2/\text{s}^2,
\]

from which we find

\[
v = -37.15 \text{ m/s}
\]
The positive solution is rejected because the ball hits the ground while falling.

### RELATIVE MOTION

![Diagram](image)

**Figure 2.7** Omer (frame $E$) and Ahmed (frame $A$) observe particle $P$. Omer is stationary while Ahmed is moving with constant speed relative to Omer.

The displacement, velocity, and acceleration of a particle are always described with respect to a specific coordinate system or frame of reference. Usually, this frame of reference is taken to be fixed, but what happen if this frame of reference is in motion with respect to another frame of reference.

Suppose that two persons want to observe the motion of a particle $P$. The first person Omer is stationary with respect to the earth, while the second person, Ahmed is moving with constant speed relative to the earth. Let the reference of frame of Omer be referred by frame $E$ (frame of the earth) and the reference of frame of Ahmed by frame $A$. The displacement of the particle with respect to the earth (Omer) is denoted by $x_{PE}$, and $x_{PA}$ is the displacement of the particle with respect to Ahmed. From Figure 2.7 we can write
where \( x_{AE} \) refers to the displacement of Ahmed with respect to the earth.

If we differentiate Equation 2.17 with respect to time, we get

\[
x_{PE} = x_{PA} + x_{AE}
\]

where \( x_{AE} \) refers to the displacement of Ahmed with respect to the earth.

If we differentiate Equation 2.17 with respect to time, we get

\[
v_{PE} = v_{PA} + v_{AE} \quad (2.18)
\]

In the last equation \( v_{PE} \) is the velocity of \( P \) relative to \( E \), \( v_{PA} \) is the velocity of \( P \) relative to \( A \), and \( v_{AE} \) is the velocity of \( A \) relative to \( E \).

**Example 2.7** A man, in a car, is driving on a straight highway at constant speed of 70 km/h relative to the earth. Suddenly, he spots a truck traveling in the same direction with constant speed of 60 km/h relative to the earth.

a) What is the velocity of the truck relative to the car?

b) What is the velocity of the car relative to the truck?

c) If the car spots the truck when they are 1.5 km apart, how long does it take the car to overtake the truck?

**Solution**

a) Let \( v_{TC} \) denotes the velocity of the truck relative to the car, and \( v_{TE} \) and \( v_{CE} \) denote, respectively, the velocities of the truck and the car relative to the earth. Now from Equation 2.18 we have

\[
v_{TE} = v_{TC} + v_{CE}
\]

or

\[
v_{TC} = v_{TE} - v_{CE} = 60 - 70 = -10 \text{ km/h}
\]

The minus sign indicates that the car is moving in the opposite direction relative to the car.

b) Again from Equation 2.18, the velocity of the car relative to the truck \( v_{CT} \) is
\[ v_{CE} = v_{CT} + v_{TE} \]

or

\[ v_{CT} = v_{CE} - v_{TE} = 70 - 60 = 10 \text{ km/h} \]

Note that \( v_{CT} = -v_{TC} \) as one should expect.

c) The time needed for the car to overpass the truck is

\[ t = \frac{1.5}{v_{CT}} = \frac{1.5}{10} = 0.15 \text{ h} = 9 \text{ minutes} \]
PROBLEMS

2.1 An automobile travels on a straight track at 80 km/h for 30 minutes. It then travels at 60 km/h for another 20 minutes.
   a) What is the average velocity of the automobile during the 50 minutes trip?
   b) What is the average speed of the automobile during the whole trip.

2.2 A particle moves according to the equation \( x = 8t^2 + 2 \), where \( x \) in meters and \( t \) in seconds.
   a) Find the average velocity during the time interval from 2 s to 4 s.
   b) Find the instantaneous velocity at \( t=2.5 \) s.

2.3 Figure 2.8 is a plot of the displacement \( x(t) \) for a particle that is initially at rest, then moves forward, and then stops.
   a) Find the average velocity of the particle during the interval \( t=1 \) s to \( t=10 \) s.
   b) Find the instantaneous velocity at \( t=4 \) s.

2.4 A particle moves along the \( x \)-axis according to \( x = 5t^2 + 30t \), where \( x \) in meters and \( t \) in seconds. Calculate
   a) the instantaneous velocity at \( t=3.0 \) s,
   b) the average acceleration during the first 3.0 s,
   c) The instantaneous acceleration at \( t=3.0 \) s.
2.5 A car traveling initially at a speed of 80 km/h is accelerated uniformly to a speed of 105 m/s in 15 s. What is the distance traveled by the car during this 15 s interval.

2.6 The initial speed of a body is 6.4 m/s. If its acceleration is 3.5 m/s$^2$, find its speed after 2.8 s.

2.7 An automobile moving with constant acceleration travels a distance of 60 m in 6 s. If its final speed is 15 m/s, 
   a) what is its acceleration?
   b) what is its initial speed?

2.8 A train starts from rest at a station and accelerates at a rate of 600 km/h$^2$ for 10 minutes. After that it travels at constant speed for 1 h, and then slows down at -800 km/h$^2$ until it stops at another station. Find the total distance covered by the train.

2.9 A car traveling at constant speed of 50 km/hr is 60 m from a stationary lorry when the driver slams on the brakes. If the car just missed the impact, 
   a) find the acceleration of the car, 
   b) find the time interval from the instant the driver slams on the brakes to the instant the car stopped.

2.10 At the instant the traffic light turns green, a car starts from rest with constant acceleration of 2 m/s$^2$. At the same instant a truck, traveling with constant speed of 10 m/s overtakes and passes the car.
   a) How far beyond the traffic light will the car overtake the truck?.
   b) What is the speed of the car at that instant.

2.11 A car traveling at a constant speed of 60 km/h passes a trooper hidden behind a tree. Two second after the car passes the tree, the trooper sets in chase after the car with constant
acceleration of 3.5 m/s\(^2\). How long does it take the trooper to overtake the car?

![Graph of motion](image)

**Figure 2.9** problem 2.12

2.12 The graph in Figure 2.9 depicts the motion of an automobile in a street.
   a) Find the displacement of the automobile during the following intervals: 0-4s, 4s-12s, and 12s-16s.
   b) Find the velocity of the automobile at t=2s, t=8s, and at t=14s.
   c) Find the acceleration of the automobile during 0-4s, 4s-12s, and 12s-16s.

2.13 A small ball is thrown vertically upward and rises to a maximum height of 20 m.
   a) With what velocity the ball is thrown.
   b) How long will it be in the air.

2.14 A small ball is released from a height of 45 m.
   a) Find the velocity of the ball just before it hits the ground.
   b) How long does it take the ball to hit the ground?
2.15 A body is thrown vertically upward from the edge of a building 21m high. In its way down it misses the building and hits the ground below with a speed of 24 m/s, as in Figure 2.10.
   a) With what velocity the body is thrown.
   b) What is the time of flight?

2.16 Refer again to Figure 2.10. Now the height of the building is 30m and the body is thrown with a speed of 16 m/s.
   a) What is the average velocity of the body during the whole flight?
   b) What is the average speed of the body during the same interval?

2.17 A bird is traveling vertically upward at a constant speed of 22.5m/s with a piece of bread in his mouth. When it is 150 m above the ground the bread piece is released from the bird’s mouth.
   a) Find the speed with which the bread hits the ground.
   b) How long does the bread take to reach the ground?
   c) What is the average speed of the bread?
   d) What is the average velocity of the bread?

2.18 A falling object starting from rest requires 1.5 s to travel the last 30m before hitting the ground. From what height above the ground did it fall.
2.19 A ball is thrown vertically upward. In its way up it passes a point P with speed $v$, and another point Q, 2.5 m higher than P with a speed $0.25v$. Find the value of $v$.

2.2 A child releases a ball from the balcony of his house that is 27 m high. The ball hits a man walking with constant speed. The man was 20 m from the point of impact when the ball was released. What was the speed of the walking man?

2.21 Two objects released from the same height 0.5 s apart. How long after the first object begins to fall will the two objects be 8.0 m apart?

2.22 A parachutist jumps horizontally to the air from a height of 2000 m and falls freely 400m before the parachute opens. The parachutist hits the ground with a speed of 5 m/s.
   a) Calculate the acceleration of the parachute while it is open.
   b) How long was the parachutist in the air.

2.23 A student in his way back to the home is traveling at constant speed of 50 km/h. What would be the velocity of the trees in the roadside as observed by the student?

2.24 A river has a steady speed of 0.2 m/s. A man swims upstream a distance of 500 m and returns to the starting point. If the man can swim at a speed of 1.0 m/s in still water, how long does the trip take?

2.25 A parachutist throws a stone vertically upward with a speed of 12 m/s relative to himself. At the instant he throws the stone he is 120 m high and falling at 4m/s. How long does it take the stone to reach the ground?