CHAPTER 5

CIRCULAR MOTION AND GRAVITATION
In the previous chapter we discussed Newton's laws of motion and its application in simple dynamics problems. In this chapter we continue our study of dynamics and the applications of Newton's laws, specially on circular motion. Newton's law of gravitation is also addressed which is the central law in planet and satellite motion.

5.1 CENTRIPETAL FORCE

In section 3.3 we have showed that, if a particle moves with constant speed $v$ in a circular path of radius $r$, it acquires a centripetal acceleration due to the change in the direction of the particle's velocity. The direction of the centripetal acceleration is always toward the center of the path and its magnitude is given by

$$a_r = \frac{v^2}{r}. \quad (5.1)$$

The subscript $r$ referring to the radial component of the acceleration. According to Newton's second law this acceleration should be a result of an applied force acting on the particle toward the center of the path, and should have a magnitude of

$$F_r = ma_r = m\frac{v^2}{r}. \quad (5.2)$$

Because of its direction such a force is called the centripetal force. Both the centripetal acceleration and the centripetal force are vector quantities whose magnitudes are constant but whose directions are always changing so as to point toward the center of the circular path.

It should be noted that any force in nature can be treated as a centripetal force if it acts on a particle in a direction toward the center of the circular path followed by the particle. The frictional
force is the centripetal force when a car rounding a curve, and the
tension is the centripetal force when you whirl a ball, tied to a
string, in a horizontal circle.

**Remark:** The word centripetal referring to a specific direction of
the force and not to a new kind of forces. It is like horizontal or
vertical.

**Example 5.1** A flat (unbanked) curve
on a highway has a radius of 100 m. If
the coefficient of static-friction between
the tires and the road is 0.2, what is the
maximum speed with which the car will
have in order to round the curve
successfully?.

**Solution** Here there are three forces acting on the car: The weight
and the normal force act perpendicular to the plane of motion, and
the static frictional force which must be parallel to the road. Hence
the centripetal force on the car is the force of static friction, so we
have

\[ f_s = \mu_s N = m \frac{v^2}{R}, \]

but, since there is no motion in the vertical direction we can write

\[ N = mg. \]

Solving the two equations for \( v \) we get

\[ v = \sqrt{\mu_s g R} = 50.4 \text{km/hr}. \]
Example 5.2  A circular curve of a road is designed for traffic moving at 60 km/hr without depending on the friction. If the radius of the curve is 80 m, what is the correct angle of banking of the road.

![Diagram of a circular curve with normal force components](image)

**Figure 5.2** Example 5.2, with the free body diagram.

**Solution**  In the banked roads, the normal force $N$ should be resolved into two components: one toward the center of the curve (horizontal), and the other vertical as shown in Figure 5.2(b). The centripetal force will be then the component $N \sin \theta$, i.e.,

$$N \sin \theta = m \frac{v^2}{R},$$

and, since there is no motion in the vertical direction we have

$$N \cos \theta = mg.$$

Substituting for $N$ from the second equation into the first equation, we have
\[ mg \tan \theta = m \frac{v^2}{R} \]

or

\[ \theta = \tan^{-1} \frac{v^2}{gR} = 19.5^\circ \]

**Example 5.3**  A ball of mass 1 kg is attached to one end of a string 1 m long and is whirled in a horizontal circle, as shown in Figure 5.3. Find the maximum speed the ball can attain without breaking the string. The breaking strength of the string is 500 N.

**Solution**  The only two forces acting on the ball are the weight and the tension. Since the weight is normal to the plane of the circle, the centripetal force in this case is the tension, so we can write

\[ T = m \frac{v^2}{R} . \]

To find the speed at the verge of breaking, we have to substitute for \( T \) by its breaking value, i.e.,

\[ v_{\text{max}} = \sqrt{\frac{\frac{T_{\text{max}} R}{m}}{1}} = \sqrt{\frac{500 \times 1}{1}} = 22.4 \text{ m/s} \]

**5.2 NONUNIFORM CIRCULAR MOTION**
In the previous section we have considered the circular motion with constant speed (uniform circular motion). When the magnitude of the velocity is not constant but rather change with time we have the nonuniform circular motion. Now what will happen if the velocity changes both in magnitude and in direction. The change in the speed will add another contribution to the acceleration. Resolving the acceleration vector into two perpendicular components: radial component and tangential component, we can write

\[
a = a_r \hat{r} + a_\theta \hat{\theta},
\]

(5.3)

where \( \hat{r} \) is a unit vector directed along the radius of the circular path, and \( \hat{\theta} \) is another unit vector tangent to the path. The radial component, \( a_r \), is the centripetal acceleration defined previously, and the tangential component, \( a_\theta \), is the new contribution due to the change in the magnitude of the particle's velocity, so we will expect

\[
a_\theta = \frac{d|v|}{dt}.
\]

(5.4)

**Remark:** In applying Newton's second law for the circular motion, the coordinate axes will be the radius-axis and the tangent-axis, so all the applied forces have to be resolved accordingly. The law now reads

\[
F_r = ma_r, \text{ and } F_\theta = ma_\theta.
\]

(5.5)

The positive senses of \( \hat{r} \) and \( \hat{\theta} \) will be chosen toward the center, and counterclockwise respectively.
Example 5.4  A small body of mass $m$ swings in a vertical circle at the end of a cord of length $L$ as shown in Figure 5.4. If the speed of the body when the cord makes an angle $\theta$ with the vertical is $v$, find

a) the radial and the tangential components of the acceleration at this point,

b) the tension in the cord at the same point.

Solution The weight have to be resolved as shown in the free-body diagram of the system. As it is clear from the diagram, the radial component is

\[ a_r = \frac{v^2}{R} = \frac{v^2}{L}, \]

and the tangential component is

\[ a_\theta = \frac{F_\theta}{m} = g \sin \theta. \]

b) Since $F_r = ma_r$, we have

\[ T - mg \cos \theta = m \frac{v^2}{L} \]

or

\[ T = m(g \cos \theta + \frac{v^2}{L}) \]
Example 5.5 A vehicle of mass 350 kg moves on a roller-coaster as shown in Figure 5.5.

a) If the speed of the vehicle at point A is 18 m/s, what is the normal force the track exerts on the vehicle?

b) What is the maximum speed for the vehicle to remain on track at point B?

Solution a) Examining the free-body diagram of the vehicle at point A we see that N is toward the center, while mg is away from the center. Applying the equation
\[ F_r = ma_r = m \frac{v^2}{r} \]

We obtain

\[ N - mg = m \frac{v^2}{r}, \text{ or} \]

\[ N = m \left( \frac{v^2}{r} + g \right) = 2.23 \times 10^3 \text{ N} \]

b) For the vehicle to be on track, the normal force must have a positive value, that is, \( N > 0 \). Now from the free body diagram of the vehicle at point b we write

\[ mg - N = m \frac{v^2}{r}, \text{ or} \]

\[ N = m \left( g - \frac{v^2}{r} \right) > 0 \]

This leads to

\[ v < \sqrt{gr} \]

So we get \( v_{\text{max}} = \sqrt{gr} = 9.39 \text{ m/s} \)
5.3 NEWTON’S LAW OF GRAVITATION

The law states that every particle in the universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. Thus the gravitational force exerted on a particle of mass \( m_1 \) by a particle of mass \( m_2 \) is

\[
F_{12} = G \frac{m_1 m_2}{r_{12}^2} \hat{r}, \quad (5.6)
\]

where \( r_{12} \) is the distance between the two particles and \( \hat{r} \) is a unit vector directed from \( m_1 \) to \( m_2 \). The universal constant \( G \) is called the gravitational constant with a value, in SI units of

\[
G = 6.672 \times 10^{-11} \text{ N.m}^2/\text{kg}^2.
\]

It can be shown that the force exerted by any homogeneous sphere is the same as if the entire mass of the sphere is concentrated at its center. Therefore, the force exerted by the earth on a small body of mass \( m \), a distance \( r \) from its center, is

\[
F = G \frac{M_e m}{r^2}, \quad r > R_e \quad (5.7)
\]

where \( M_e \) and \( R_e \) are the earth’s mass and the earth’s radius, respectively. This force is directed toward the center of the earth. Inside the earth, the force would decrease as approaching the center rather than increasing as \( \frac{1}{r^2} \). At the center of the earth the gravitational force on the body would be zero, why?

For freely falling body the only force acting is the gravitational force of the earth and the acceleration produced is the
acceleration due to gravity, \( g \). Now, from Newton’s second law, and assuming the body to be at the surface of the earth, we have

\[
\sum F = G \frac{M_e m}{R_e^2} = mg, \quad (5.8)
\]

or

\[
g = G \frac{M_e}{R_e^2}. \quad (5.9)
\]

The mass of the earth can be calculated using Equation (5.9) as

\[
M_e = \frac{R_e^2 g}{G} = 5.96 \times 10^{24} \text{ kg},
\]

with \( R_e = 6370 \text{ km} \)

The force acting on a particle at a distance \( h \) above the earth’s surface is, from Equation (5.6) and Equation (5.8)

\[
F = G \frac{M_e m}{(R_e + h)^2} = mg', \quad (5.10)
\]

or

\[
g' = G \frac{M_e}{(R_e + h)^2}. \quad (5.11)
\]

Therefore, \( g' \) decrease with increasing altitude.

**Example 5.6** Two bodies of mass 60 kg, and 80 kg are placed 2 m apart. Calculate the gravitational force exerted by one body on the other.
Solution From Equation (5.5) we have

\[
F = G \frac{m_1 m_2}{r^2} = \left(6.67 \times 10^{-11}\right) \frac{(60)(80)}{(2)^2} = 8 \times 10^{-8} \text{ N}.
\]

Example 5.7 Three bodies of mass 2 kg, 4 kg, and 6 kg are arranged as shown in Figure 5.6. Calculate the total force acting on the 2-kg mass by the other two masses.

Solution The force exerted on the 2-kg mass by the 4-kg mass is

\[
F_{24} = G \frac{m_2 m_4}{r_{24}^2} \mathbf{i} = \left(6.67 \times 10^{-11}\right) \frac{2 \times 4}{(2)^2} \mathbf{i} = 1.33 \times 10^{-10} \mathbf{i} \text{ N},
\]

and the force exerted by the 6-kg is

\[
F_{26} = G \frac{m_2 m_6}{r_{26}^2} \mathbf{j} = \left(6.67 \times 10^{-11}\right) \frac{2 \times 6}{(1)^2} \mathbf{j} = 8.0 \times 10^{-10} \mathbf{j} \text{ N}.
\]

Therefore, the total force acting on the 2-kg mass due to the 4-kg and the 6-kg masses is the vector sum of \(F_{24}\) and \(F_{26}\):

\[
F_2 = F_{24} + F_{26} = (1.33\mathbf{i} + 8.0\mathbf{j}) \times 10^{-10} \text{ N}
\]
Example 5.8 Calculate the magnitude of the acceleration due to gravity at an altitude of 100 km.

Solution From Equation (5.10) we have

\[
g' = G \frac{M_e}{(R_e + h)^2} = \left(6.67 \times 10^{-11}\right)\left(5.98 \times 10^{24}\right) \left(6.37 \times 10^6 + 1 \times 10^3\right)^2 \\
= 9.5 \text{ m/s}^2.
\]

This means that \(g'\) is decreased by a 3%.

5.4 SATELLITE MOTION

From section 3.2 we show that if one launch a projectile from the surface of the earth with a rather small velocity, the trajectory will be a parabola provided that the air resistance is neglected. By increasing the velocity of projection we can increase the size of the trajectory, and above a certain critical value of the velocity, the trajectory will miss the earth and the projectile has become an earth satellite. In this case the force acting on the projectile is no longer constant but varies inversely as \(\frac{1}{r^2}\), with \(r\) is the radius of the satellite’s orbit. It turns out that under such an attractive force, the projectile’s path may be a circle, an ellipse, a parabola, or a hyperbola. The circular path will be considered for simplicity.

We have learned that a particle in a uniform circular motion has a centripetal acceleration given by \(a_r = \frac{v^2}{r}\), with \(v\) is its speed
and $r$ is the radius of the circular path. In satellite motion the gravitational force (Equation 5.7) is the force that provides such acceleration, that is

$$G \frac{M_e m}{r^2} = m \frac{v^2}{r},$$

where $m$ is the mass of the satellite and $r$ is the radius of the satellite orbit. Solving for $v$ we get

$$v = \sqrt{\frac{GM_e}{r}}. \quad (5.12)$$

The period of revolution is

$$\tau = \frac{2\pi r}{v}.$$

By substituting for $v$ from Equation 5.12, we get

$$\tau = 2\pi \sqrt{\frac{r^3}{GM_e}} = \frac{2\pi r^{3/2}}{\sqrt{GM_e}}. \quad (5.13)$$

It should be clear that the previous considerations are also applicable to the motion of our moon around the earth and the motion of the planets around the sun.

**Example 5.9** If one want to place a communication satellite into a circular orbit of radius 6800 km. What must be its speed, and its period?

**Solution** From Equation (5.11) we obtain
\[ v = \sqrt{\frac{GM_e}{r}} \]
\[ = \sqrt{\frac{\left(6.67 \times 10^{-11}\right) \left(5.98 \times 10^{24}\right)}{6.8 \times 10^6}} \]
\[ = 7.66 \times 10^3 \text{ m/s}. \]

The period is, from Equation (5.13),

\[ \tau = \frac{2\pi r^{3/2}}{\sqrt{GM_e}} = \frac{2\pi \left(6.8 \times 10^6\right)^{3/2}}{\sqrt{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}} \]
\[ = 1.55 \text{ hr}. \]

5.6 **KEPLER’S LAWS**

The fact that the planets move about the sun in such a way that the areal velocities are constant was found by Johannes Kepler in 1609. Keplere studied the data of his teacher Tycho Brahe and eventually formulated the following three laws applied to the solar system.

1. **The law of orbits:** Each planet moves in an ellipse with the sun as a focus.
2. **The law of areas:** The radius vector drawn from the sun to any planet sweeps out equal areas in equal times.
3. **The law of periods:** The square of the period of revolution of any planet is proportional to the cube of the major axis of the orbit.
CHAPTER 5 CIRCULAR MOTION AND GRAVITATION

PROBLEMS

5.1 A 2-kg mass moves in a circle with a speed of 4 m/s. If the radius of the circle is 0.5 m, what is the centripetal force acting on the mass?

5.2 A 2-kg mass is attached to a light string rotates in circular motion on a horizontal, frictionless table. The radius of the circle is 1.0 m, and the string can support a maximum force of 240 N. What is the maximum speed the mass can have before the string breaks?

5.3 A car rounds an unbanked curve with a radius of 50 m. If the coefficient of static friction between the tires and the road is 0.6, what is the maximum speed the car can have in order not to slide during the rounding.

5.4 A 200-g mass on a frictionless table is attached to a hanging block of mass 800 g by a cord through a hole in the table as in Figure 5.7. The suspended block remains in equilibrium while the mass revolves on the surface of the table in a circle of radius 0.6 m.

a) What is the tension in the cord?

b) What is the speed of the mass?

c) What is the centripetal force acting on the mass?

5.5 A car moving at 50 km/hr want to turn a 15°-banked curve with radius of 40 m on a rainy day (friction is neglected). Would the car make the turn successfully? If not with what speed must it move?
5.6 A small mass is placed 0.5 m from the center of a rotating, horizontal table that rotates with a constant speed of 1.5 m/s. The mass is in the verge of slipping with respect to the turntable. What is the coefficient of static friction between the mass and the table?

5.7 A car travels over a hill, which can be regarded as an arc of a circle of radius 150 m, as in Figure 5.8. What is the maximum speed the car can have without leaving the road at the top of the hill?

5.8 A coin is placed inside an open basket, which in turn allowed to rotate in a vertical circle of radius 1.2 m. What is the minimum speed of the basket at the top of the circle if the coin not to fall off?

5.9 A 30-kg child sits in a conventional swing of length 2.5 m. The tension in each chain that support the seat of the swing at the lowest point is 300 N, and the mass of the seat is 4 kg.
   a) What is the child’s speed at the lowest point?
   b) What is the force acting on the child by the seat?

5.10 A 0.2-kg pendulum bob passes through the lowest point with a speed of 6 m/s. What is the tension in the cord of the pendulum if it is 1 m long?

5.11 A small ball of mass $m$ is suspended from a string of length $L$ that makes an angle $\theta$ with the vertical. The ball revolves in a horizontal circle with constant speed (conical pendulum), as in Figure 5.9. Find the speed of the ball.
5.12 An object tied to the end of a string is whirled in a vertical circle of radius $R$. What is the minimum speed below which the string would become loose at the highest point?

5.13 A Ferris wheel (أرجوحة الدوّلاب) with radius 20 m rotates at 8 m/s. Find the apparent weight of a 40-kg boy at

- a) the top of the Ferris wheel,
- b) the bottom of wheel.

5.14 A skater moves on an irregular track as shown in Figure 5.10. Point $p$ in the track is at the top of an arc of a circle of radius 20m. What is the maximum speed of the skater at point $p$ to remain in the track?
5.15 A person enters a Rotor of radius 3 m, as shown in Figure 5.11. If the coefficient of static friction between the person and the wall is 0.4, what is the minimum speed with which the Rotor must rotate such that the person is safe from falling?

5.16 A small block of mass \( m \) is placed inside a cone that is rotating about its axis, as in Figure 5.12. If the inside wall of the cone is frictionless, what is the speed of the cone to keep the mass from sliding down?

5.17 A 3-kg mass is connected to a vertical rod by means of two massless strings, as in Figure 5.13. The strings are taut, and form two sides of equilateral triangle of length 2 m. The rod rotates about its axis and the mass rotates in a horizontal plane. If the tension in the upper string is 120 N,

a) draw the free-body diagram of the mass,

b) calculate the tension in the lower string,

c) calculate the speed of the mass.

5.18 A distance of 0.5 m separates two particles of mass 200 kg, and 500 kg. What is the gravitational force
exerted by one particle on the other?

**5.19** Four identical balls each of mass 20 kg are located at the corners of a square of side 1 m, as in Figure 5.13. Calculate the total force acting on one ball from the other three balls.

**5.20** What would be the weight of a 90-kg man at the top of a hill of height 500 km?

**5.21** A satellite of mass 400 kg is in a circular orbit of radius \(6.1 \times 10^6\) m about the earth. Calculate
   a) The period of its revolution,
   b) The gravitational force acting on it.