6.1 WORK

Consider an object displaced a distance $s$ under the action of the constant force $F$ as shown in Figure 6.1. The work done by this force is defined as the product of the magnitude of the displacement and the component of the force in the direction of the displacement. Since the component of $F$ in the direction of $s$ is $F \cos \theta$, the work $W$ done by $F$ is given by

$$W = (F \cos \theta) s$$  \hspace{1cm} (6.1)

Comparing Equation 6.1 with definition of the scalar product the of two vectors, we conclude that the work done by a constant force can be expressed as

$$W = F \cdot S$$  \hspace{1cm} (6.2)

From this definition, we see that work is done by $F$ on an object if the object undergoes a displacement and $F$ has a nonzero component in the direction of $s$. For example, if you push hard against a wall, no work is done by you on the wall even though you tire. Also the work done by a force is zero when the force is perpendicular to the displacement. Thus, we find that the meaning of work in physics is different from its common meaning in day-to-day affairs.

The work may be positive, or negative depending on the direction of $F$ relative to $s$. It is positive if $F \cos \theta$ is in the direction of $s$, and is negative if $F \cos \theta$ is in the opposite direction of $s$. 
An example of the negative work is the work done by a frictional force as a body slides along a rough surface (Figure 6.2). Since the angle $\theta$ between the frictional force $f$ and the displacement $s$ is $\theta = \pi$, the work done by this frictional force is given by

$$W_f = f \cdot s = f s \cos \pi = -f s$$  \hspace{1cm} (6.3)

It is clear from Figure 6.2 that the normal force $N$ and the weight $mg$ do no works since these two force are perpendicular to the displacement, that is $\theta = \pi/2$.

**Remark:** Since the frictional force is in opposite direction with the displacement, (Figure 6.2), the work done by any frictional force is always negative.

From Equation 6.2 we conclude that the work is a scalar quantity. Its unit is force multiplied by length. Therefore, the SI units of work is Newton meter (N.m) or Joule (J), while the cgs unit is erg.

**Example 6.1** A block of mass 2 kg moves under the influence of a force $F = 20$ N, which makes an angle of $37^\circ$ above the horizontal. The block moved a distance $s = 4$ m, on a rough surface of $\mu = 0.2$, as shown in Figure 6.3. Calculate,

a) the work done by $F$
b) the work done by friction.
c) the net work done on the block.

**Solution**

a) The work done by the force $F$ is, from Equation 6.2

\[ W_F = F_s \cos \theta \]
\[ = (20)(4)(\cos 37^\circ) = 63.9 \text{ J.} \]

b) Using equation 6.3, the work done by the force of friction is

\[ W_f = -f_s = -\mu N_s \]

but

\[ N = mg - F \sin \theta \] (why?)

so

\[ W_f = -0.2 \times (19.6 - 12.0) \times 4 = -6.1 \text{ J} \]

c) Since the weight ($mg$) and the normal force ($N$) do not do any work (why?), the net work $W_{\text{net}}$ is then

\[ W_{\text{net}} = W_F + W_f = 63.9 - 6.1 = 57.8 \text{ J.} \]

**6.2 WORK DONE BY A VARYING FORCE**

Consider an object being displaced along the $x$-axis under the action of a force that varies with respect to the position $x$. To find the work done by such a force as the object moves from an initial point $x_i$ to a final point $x_f$, we cannot use Equation 6.2 because it applies only for a constant force. To manage such a situation we divide the displacement into small intervals $\Delta x$ such that the force can be considered constant over such intervals. The work done by the force over this small displacement can be expressed as
\[ \Delta W = F_x \Delta x \]  

6.4

The total work done along the displacement from \( x_i \) to \( x_f \) is the sum of many such terms:

\[ W = \sum \Delta W = \sum_{x_i}^{x_f} F_x \Delta x \]  

6.5

If the displacement intervals are allowed to approach zero, then the number of terms in the sum increases infinitely. This limit of the sum is called the integral and is represented by

\[ W = \lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F_x \Delta x = \int_{x_i}^{x_f} F_x dx \]  

6.6

In general we can express the work done by any force for the displacement from initial point \( i \) to a final point \( f \) as

\[ W = \int_{i}^{f} F \cdot ds \]  

6.7

Note that if the force \( F \) is constant over the displacement, we recover Equation 6.2.

### 6.3 Work done by a spring

As an example of force varies with position we consider here the force of a spring. Figure 6.4(a) shows a spring with one end is fixed, while the other end is attached to a block. The spring is in its equilibrium state, that is, neither compressed nor extended. In Figure 6.4(b) the block is displaced to the right and the spring is now stretched a distance \( x \). If the block is displaced to the left as in
Figure 6.4(c) the spring is now compressed. Again the spring will exert a force on the body toward its equilibrium position. The magnitude of the force in both cases is given by 

**Hook’s law:**

\[ F_s = -kx \]

(6.8)

where \( x \) is the displacement of the body from its equilibrium position \((x=0)\). The force constant \( k \) is a measure of the stiffness of the spring. The minus sign in Equation (6.8) tells that the force of the spring is always opposes the displacement. Let us calculate the work done by the force \( F_s \), as the body moves from an initial position \( x_i \) to a final position \( x_f \). Applying Equation (6.7), we get

\[
W_s = \int_{x_i}^{x_f} F_s \, dx = \int_{x_i}^{x_f} -k \, dx = -k \int_{x_i}^{x_f} x \, dx
\]

, or

\[
W_s = \frac{1}{2} kx_i^2 - \frac{1}{2} kx_f^2.
\]

(6.9)

The work done by a spring, \( W_s \), can be positive or negative depending on whether the mass moves toward or a way from the equilibrium position:
1- If the mass moves toward the equilibrium position then \( x_f = 0 \) and, from Equation (6.9), the work is positive. This is due to the fact that the force and the displacement, in this case, are in the same direction.

2- If the mass moves away from the equilibrium position then \( x_i = 0 \), and the work is negative. Here the force and the displacement are in opposite directions.

The work done by an external force in compressing or stretching a spring is equal to the negative of the work done by the spring’s force during the corresponding displacement.

**Example 6.2:** A block is tied to a spring with force constant of 80 N/m as shown in Figure 6.4. The spring is compressed a distance 3 cm from equilibrium position.

a) Calculate the work done by the spring as the block moves from its equilibrium to its compressed position.

b) Calculate the work done by the spring as the block returns to its equilibrium position.

**Solution:**

a) The block was at its equilibrium position \( x_i = 0 \) and moves to a final position \( x_f = -3 \text{cm} = -0.03 \text{m} \). The work done during this interval is, from Equation 6.9 as

\[
W_s = 0 - \frac{1}{2} \times 80 \times (-3 \times 10^{-2})^2 = -3.6 \times 10^{-2} \text{J}
\]

b) Now \( x_i = -3 \text{cm} = -0.03 \text{m} \) and \( x_f = 0 \). So from Equation 6.9 we have

\[
W_s = \frac{1}{2} \times 80 \times (-3 \times 10^{-2})^2 - 0 = 3.6 \times 10^{-2} \text{J}.
\]

### 6.4 WORK-KINETIC ENERGY THEOREM
Let us consider that the net force acting on an object is in $x$ direction, then equation (6.7) takes the form

$$W_{net} = \int_{i}^{f} F_x \, dx$$

(6.10)

where $F_x$ is the net force acting on the body in the $x$ direction. Newton’s second law states that $F_x = ma_x$, and the acceleration $a$ can be expressed as

$$a = \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = \frac{dv}{dx} v$$

Substituting this in Equation (6.6), we have

$$W_{net} = \int_{i}^{f} m\frac{dv}{dx} \, dx = \int_{v_i}^{v_f} mv \, dv = \frac{1}{2}mv^2 \bigg|_{v_i}^{v_f}$$

$$= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

(6.11)

The quantity $K = \frac{1}{2}mv^2$ is called the kinetic energy of a particle of mass $m$ and speed $v$. It is convenient to write Equation 6.11 as

$$W_{net} = K_f - K_i = \Delta K$$

(6.12)

That is, the net work done by the force $F_x$ along $x$-direction from $x = x_i$, to $x = x_f$ is given by the change in kinetic energy, $\Delta K$. 
Example 6.3: Consider the problem of example (6.1). If the initial velocity is zero, find the final velocity after the block moves a distance 4 m.

Solution: From example (6.1) the net work is given by

$$W_{\text{net}} = 57.8 \text{ J}.$$ 

Applying the work-energy theorem with $v_i=0$, we have

$$W_{\text{net}} = \frac{1}{2}mv_f^2$$

or

$$v_f^2 = \frac{2W_{\text{net}}}{m} = \frac{2 \times 57.8}{2} = 57.8 \text{ m}^2/\text{s}^2,$$

so

$$v_f = 7.6 \text{ m/s}.$$ 

Example 6.4  A mass of 2 kg is pushed up a rough inclined plane by a force $F=20 \text{ N}$. The mass is displaced a distance 0.5 m on the inclined plane. Calculate,

a) the work done by the force of gravity,

b) the work done by the force $F=20 \text{ N}$,

c) the work done by the force of friction if $\mu_k=0.2$,

d) If the mass has a kinetic energy of 1.2 J at the beginning of the displacement, what is the kinetic energy at the end of the displacement?

Solution: a) Since the force of gravity is downward, the work done by the force of gravity is given by
\[ W_g = - (mg \sin 37^\circ) d = -(2 \times 9.8 \times 0.6) \frac{1}{2} = -5.88 \text{ J} \]

b) \( \mathbf{F} \) is in the same direction as the displacement \( \mathbf{s} \), so the work done by \( \mathbf{F} \) is,

\[ W_F = F \cdot s = 20 \times 0.5 = 10 \text{ J}. \]

c) The force of friction is given by

\[ f = \mu N = \mu mg \cos 37^\circ = 0.2 \times 2 \times 9.8 \times \cos 37^\circ = 3.13 \text{ N} \]

Now the work done by this frictional force is

\[ W_f = -f \cdot s = -3.13 \times 0.5 = -1.57 \text{ J} \]

d) Using the work-kinetic energy theorem we have

\[ \Delta K = W_{\text{net}} = W_g + W_F + W_f \]
\[ = -5.88 + 10 - 1.57 = 2.55 \text{ J}. \]
**But** \[ \Delta K = K_f - K_i. \] So \[ K_f = \Delta K + K_i = 2.55 + 1.2 = 3.75 \text{ J} \]

### 6.5 POWER

The power is defined as time rate at which work is done. **The average power** during the time interval \( \Delta t \) is defined as

\[
\bar{P} = \frac{\Delta W}{\Delta t} \quad (6.13)
\]

The instantaneous power is the limiting value of \( \bar{P} \), that is

\[
P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} \quad (6.14)
\]

If \( F \) is constant, then \( dW = F \cdot ds \), and the power (6.14) becomes

\[
P = \frac{dW}{dt} = F \cdot \frac{ds}{dt} = F \cdot v, \quad (6.15)
\]

where the velocity \( v \) is defined by \( \frac{ds}{dt} \). The unit of power is J/s or watt, (W), with 1W=1 J/s=1 kg.m²/s³.

**Example 6.5:** A 1500-kg car accelerates uniformly from rest to a speed of 10 m/s in 3 s. Find

a) the work done on the car in this time,

b) the average power delivered by the engine in the first 3 s,

c) the instantaneous power delivered by the engine at \( t=2 \text{ s} \).

**Solution:** After 3 s, \( v_f = 10 \text{ m/s}, m = 1500 \text{ kg} \)
a) the work done is given by

\[ W = \frac{1}{2} m v_f^2 - 0 \]

\[ = \frac{1}{2} \times 1500 \times (10)^2 = 7.50 \times 10^4 \text{ J} , \]

b) \[ P = \frac{7.5 \times 10^4}{3} = 2.5 \times 10^4 \text{ W}, \]

c) let us find the acceleration \( a \), from \( v = v_o + at \), we have,

\[ a = \frac{v - v_o}{t} = \frac{10 - 0}{3} = 3.33 \text{ m/s}^2. \]

The velocity at \( t=2 \text{s} \), is then

\[ v = 0 + 3.33 \times 2 = 6.66 \text{ m/s} \]

and the force of the engine is calculated using Newton's second law

\[ F = ma = 1500 \times 3.33 = 5000 \text{ N}. \]

Now, the instantaneous power is

\[ P = Fv = 5000 \times 6.66 = 3.33 \times 10^4 \text{ W}. \]
**CHAPTER 6 WORK AND ENERGY**

**PROBLEMS**

6.1 A block is pushed 10 m along a horizontal surface by a 3-N horizontal force. The frictional force on the block is 1.5 N.
   a) How much work is done by the 3-N force?
   b) How much work is done by the frictional force?

6.2 A man pushes a 10-kg block 10 m, along a rough, horizontal surface with a 40-N force directed 37° below the horizontal. If the coefficient of kinetic friction is 0.2, calculate the total work done on the block.

6.3 A block of mass 20-kg is pushed by a horizontal force of 60 N on a rough horizontal surface a distance 4 m. If the block moves with constant speed, calculate
   a) the work done by the 60 N force,
   b) the work done by a frictional force,
   c) the coefficient of kinetic friction.

6.4 A particle of mass 0.5 kg is released from rest at point A inside a rough hemispherical bowl of radius 1 m, as shown in Figure 6.6. If the particle can reach a maximum height of 0.8 m (point B), calculate
   a) the work done by gravity,
   b) The work done by friction.

6.5 A block of mass $m = 12$ kg is drawn at constant speed a distance $s=20$ m along a horizontal floor by a rope exerting a
constant force \( F = 40 \text{ N} \) making an angle of 53° above the horizontal. Compute,

a) the total work done on the block
b) the work done by the rope on the block
c) the work done by friction on the block
d) the coefficient of kinetic friction between block and the floor.

6.6 A block of mass 50 kg is attached to a cord that is wrapped around a fixed pulley as shown in Figure 6.7. The block is lowered at constant acceleration of 2.5 m/s². When the block has fallen a distance 4 m, find

a) The work done by the cord,
b) the work done by the weight of the block.

6.7 A Force \( F_x = (10x + 2) \text{ N} \), where \( x \) is in m, is acting on a body from \( x = 0 \) to \( x = 2 \text{ m} \). Find the net work done by this force as the body moves from \( x = 0 \) to \( x = 2 \text{ m} \).

6.8 A block of mass 30 kg is hung vertically on a light spring with spring constant \( k \). After permanently coming to rest, the spring stretches a distance of \( x = 10 \text{ cm} \).

a) What is the work done by gravity during this motion?
b) What is the work done by the spring during this motion?
c) What is the net work done on the block?
d) Is the answer of (c) is surprising?

6.9 A force \( \mathbf{F} = (-5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \text{ N} \) acts on a particle which is displaced \( \mathbf{s} = (-2\mathbf{i} - 2\mathbf{k}) \text{ m} \).

a) Find the work done by the force on the particle.
b) Find the angle between $\mathbf{F}$ and $\mathbf{s}$.

6.10 A force vector $\mathbf{F}$ is 2 N in magnitude, and points in the positive $y$ direction. The displacement vector $\mathbf{s}$ has a negative $x$-component of 5 m, a positive $y$-component of 3 m, and no $z$-component. Find,
   a) the work done by the force $\mathbf{F}$,
   b) the angle between $\mathbf{F}$ and $\mathbf{s}$.

6.11 A small ball of mass 1 kg is attached to a light, 2 m long string. The ball is released from the horizontal position $A$, as shown in Figure 6.7.
   a) Find the work done by gravity as the particle moves from point $A$ to point $B$, the lowest position of the ball.
   b) Find the speed of the particle at point $B$.

6.12 A car of mass 2000 kg is pushed from rest to a speed $v$, a distance 20 m. The work done on the car is 6000 J, find
   a) the final speed $v$ of the car,
   b) the horizontal force exerted on the car, assuming it is constant.

6.13 A 4-kg mass has an initial velocity $\mathbf{v} = (-3\mathbf{i} - \mathbf{j})$ m/s.
   a) Find the kinetic energy at this time.
   b) Find the change in its kinetic energy if its velocity changes to $(4\mathbf{i}+2\mathbf{j})$ m/s.

6.14 A 5-kg body is subject to a force that varies with position as shown in Figure 6.8. The body starts from rest at $x=0$. Find the speed of the body at
6.15 A 3-kg block is attached to a light spring of spring constant $k = 400$ N/m. The block is displaced to the right 4 cm, on a smooth surface, from its equilibrium position and released from rest. Find the maximum speed (at $x=0$).

6.16 A 3-kg block starts with initial speed of 8 m/s at the bottom of an inclined plane of inclination angle 20°. The frictional force that retards its motion is 15 N.

a) If the block is directed up the incline, how far will it move before it stop?
b) Will it slide back down the incline?

6.17 A car of mass 1000 kg is pulled along a rough surface with coefficient of friction $\mu_k = 0.3$.

a) How much power must the engine deliver to move the car at constant speed of 40 m/s.
b) How much work does the engine in 2 minutes do?

6.18 A car engine delivers a power of $3 \times 10^4$ W when moving with constant speed of 20 m/s. Find the resistive force acting on the car at this speed.

6.19 A boat moves at a constant speed of 20 m/s. If the resistive force of the water is 10 N, how much power is produced by the motor?

6.20 A small block of mass 0.25 kg is pushed against a spring of force constant 800 N/m, compressing it a distance of 0.1 m.
m. When released, the block travels along a smooth circular track of radius 0.4 m, as shown in Figure 6.9.

a) Calculate the net work acting on the block in going from point A to point B (the top of the track).

b) Find the force the track exerts on the block at point B.