CHAPTER 8

SYSTEMS OF PARTICLES
8.1 CENTER OF MASS

The center of mass of a system of particles or a rigid body is the point at which all of the mass are considered to be concentrated there and all external forces were applied there. In this section we want to know how to determine the center of mass of a system.

Consider a system of two masses $m_1$ and $m_2$ located along the $x$-axis as shown in Figure 8.1. The position $x_{cm}$ of the center of mass of these two masses is defined to be

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$  \hspace{1cm} (8.1)

If $m_1 = m_2$, $x_1 = 0$, and $x_2 = d$, we find that $x_{cm} = \frac{d}{2}$, i.e., the center of mass lies midway between the two masses.

For a system of $n$-particles $m_1$, $m_2$, ..., $m_n$, the center of mass is

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n}.$$  \hspace{1cm} (8.2)

Figure 8.1 A system of two particles $m_1$ and $m_2$. The point labeled cm is the position of the center of mass of the system.
\[ x_{cm} = \frac{1}{M} \sum_{i=1}^{n} m_{i}x_{i} \quad (8.3) \]

Where \( M = m_{1} + m_{2} + \cdots + m_{n} \) is the total mass of the system.

If the system of particles is distributed on three dimension, the \( y \) and the \( z \) coordinates of the center of mass are similarly defined by

\[ y_{cm} = \frac{1}{M} \sum_{i=1}^{n} m_{i}y_{i} \quad , \quad (8.4) \]

and

\[ z_{cm} = \frac{1}{M} \sum_{i=1}^{n} m_{i}z_{i} \quad , \quad (8.5) \]

In vector notation, the position vector of the center of mass \( \mathbf{r}_{cm} \) can be expressed as

\[ \mathbf{r}_{cm} = x_{cm}\mathbf{i} + y_{cm}\mathbf{j} + z_{cm}\mathbf{k} \quad (8.6) \]

Or

\[ \mathbf{r}_{cm} = \frac{1}{M} \sum_{i=1}^{n} m_{i}\mathbf{r}_{i} \quad (8.7) \]

To find the center mass of a rigid body (continuous mass distribution) we treat the body as consisting of so large number of small elements \( dm \) such that the sums of Equations 8.3-8.5 become integrals and the coordinates of the center of mass become
The vector position of the center of mass of a rigid body is expressed as

\[ \mathbf{r}_{\text{cm}} = \frac{1}{M} \int \mathbf{r} \, dm. \]  

(8.11)

The integrals are to be evaluated over all the mass distribution of the object.

**Example 8.1** Three particles of masses \( m_1 = 1 \text{ kg} \), \( m_2 = 2 \text{ kg} \), and \( m_3 = 3 \text{ kg} \) are located as shown in Figure 8.10. Find the center of mass of this system.

**Solution** The \( x \)-component of the center of mass is

\[
x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}
= \frac{1 \times 0 + 2 \times 1 + 3 \times 4}{1 + 2 + 3}
= \frac{14}{6} = 2.33 \text{ m}.
\]
The \( y \)-component is

\[
y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{1\times2 + 2\times0 + 3\times0}{1 + 2 + 3} = \frac{2}{6} = 0.33 \text{m}
\]

The position of the center of mass is therefore

\[
r_{cm} = 2.33 \hat{i} + 0.33 \hat{j} \text{ m}
\]

**Example 8.2** Show that the center of mass of a uniform rod of mass \( M \) and length \( L \) lies midway between its ends.

**Solution** Let the rod be located along the \( x \)-axis as shown in Figure 8.3. By symmetry it is obvious that \( y_{cm} = z_{cm} = 0 \). Let us take a small element of mass \( dm \) and length \( dx \). From Equation 8.8, we have

\[
x_{cm} = \frac{1}{M} \int x \, dm
\]

To solve the integral we need to find a relation between the mass \( dm \) and the variable \( x \). To find such a relation we define the linear mass density \( \lambda \) (mass per unit length), as \( \lambda = \frac{dm}{dx} = \frac{M}{L} \). Now the above equation becomes
\[ x_{cm} = \frac{\lambda}{M} \int_0^L x \, dx = \frac{\lambda}{M} \frac{x^2}{2} \bigg|_0^L = \frac{\lambda L^2}{2M} \]

Substituting for \( \lambda = \frac{M}{L} \), we get

\[ x_{cm} = \left( \frac{M}{L} \right) \frac{L^2}{2M} = \frac{L}{2}. \]

### 8.2 DYNAMICS OF A SYSTEM OF PARTICLES

From Equation 8.7 we have

\[ M \mathbf{r}_{cm} = \sum_{i=1}^{n} m_i \mathbf{r}_i \]

Differentiating the above equation with respect to time gives

\[ M \frac{d(\mathbf{r}_{cm})}{dt} = \sum_{i=1}^{n} m_i \frac{d(\mathbf{r}_i)}{dt} \quad (8.12) \]

Knowing that \( \frac{d(\mathbf{r}_{cm})}{dt} \) is the velocity of the center of mass and \( \frac{d(\mathbf{r}_i)}{dt} \) is the velocity of the \( i^{th} \) particle, Equation 8.12 becomes

\[ M \mathbf{v}_{cm} = \sum_{i=1}^{n} m_i \mathbf{v}_i \quad (8.13) \]

Differentiating Equation 8.13 with respect to time leads to
\[ Ma_{cm} = \sum_{i=1}^{n} m_i a_i \]  
(8.14)

Where \( \frac{d(v_{cm})}{dt} \) is the acceleration of the center of mass and \( \frac{d(v_i)}{dt} \) is the acceleration of the \( i^{th} \) particle.

From Newton's second law we know that \( m_i a_i \) represents the resultant force \( F_i \) that acts on the \( i^{th} \) particle. Thus we can write Equation 8.14 as

\[ Ma_{cm} = \sum_{i=1}^{n} F_i \]  
(8.15)

Remember that \( F_i \) is the vector sum of the external forces acting on the \( i^{th} \) particle and the internal forces resulting from the other particles of the system. From Newton's third law the internal forces form action-reaction pairs so that they cancel out in the sum of Equation 8.15. So, the right hand side of Equation 8.15 is the vector sum of all the external forces \( F_{ext} \) that act on the system. Equation 8.15 then reduces to

\[ Ma_{cm} = \sum F_{ext} \]  
(8.16)

Equation 8.16, like any vector equation can written as three equations corresponding to the components of \( a_{cm} \) and \( F_{ext} \) along the coordinates axes, that is,
\[ \sum F_x^{\text{ext}} = Ma_x^{\text{cm}} \]
\[ \sum F_y^{\text{ext}} = Ma_y^{\text{cm}} \]
\[ \sum F_z^{\text{ext}} = Ma_z^{\text{cm}} \]  \hspace{1cm} (8.17)

Equation 8.16 tells that if no net external force acting on a system, the acceleration of its center of mass is zero and thus the velocity of the center of mass of the system remains unchanged.

**Example 8.3** A shell is fired with an initial speed \( v_0 \) at an angle of \( \theta \) above the horizontal. At the top of the trajectory, the shell explodes into two equal fragments. One fragment, whose speed immediately after explosion is zero, falls vertically down, as shown in Figure 8.4. How far from the initial point does the other fragment land.

**Solution**
As the forces due to the explosion is internal, they do not affect the motion of the center of mass. Since the only external force acting on the system is the force of gravity, the center of mass follows a parabolic path (the dotted path shown in Figure 8.4) as the projectile did not explode. From Example 3.2 we obtain
\[ x_{cm} = \frac{v_o^2 \sin 2\theta}{g} \]

But from Equation 8.3 we have

\[ x_{cm} = \frac{m_1 x_1 + m_2 x_2}{M} \]

Knowing that \( x_1 = \frac{1}{2} x_{cm} \) and \( m_1 = m_2 = \frac{1}{2} M \) we get

\[ x_{cm} = \frac{\frac{1}{2} x_{cm} + x_2}{2} \]

or

\[ x_2 = \frac{2}{2} x_{cm} = \frac{3}{2} \frac{v_o^2 \sin 2\theta}{g} \]

**Example 8.4** A railroad car of mass \( M \) can move along a smooth horizontal track. A man of mass \( m \) is, initially standing at one end of the car, which is initially at rest, as shown in Figure 8.5. If the man starts to walk toward the other end of the car, describe the motion of the car.

**Solution** There is no external force acting on the man-car system along the horizontal direction. This means that the velocity of the center of mass of the system will not change and must remain zero, and so the position of the center of mass of the man-car system is the same before and after the man starts to walk.

To maintain the position of the center of mass unchanged, the car will move in a direction opposite to the direction of the walking man, as it clear from Figure 8.5.
Let us now study the motion of the car analytically. Let $x_1$ be the position of right end of the car (at which the man is initially stands) relative to a fixed axis, and $x_2$ is the new position of the same end. Assuming that the car is uniform and the man walks a distance $L$ between its ends, the position of the center of mass of the man-car system when the man is at the right end is

$$x_{cm} = \frac{mx_1 + M(x_1 + \frac{1}{2}L)}{m + M}$$

When the man is now at the left end of the car the position of the center of mass is

Figure 8.5 Example 8.4.
Equating the above two equations we obtain

\[ x_2 = x_1 - \frac{m}{M + m} L \]

The last equation tells that if the man moves a distance \( L \) to the left, the car will move to the right a distance \( mL/(M + m) \).

### 8.3 LINEAR MOMENTUM

The linear momentum \( p \) of a particle of mass \( m \) moving with velocity \( v \) is defined as

\[
p = mv
\]  \hspace{1cm} (8.18)

In SI unit system the momentum has a unit of kg.m/s. The components of momentum \( p \) in three dimensions are

\[
p_x = mv_x \quad p_y = mv_y \quad p_z = mv_z.
\]

Let us now find the relationship between the linear momentum and the force. From Newton's second law we have

\[
F = ma = m \frac{dv}{dt} = \frac{d(mv)}{dt},
\]  \hspace{1cm} (8.19)

where \( m \) is constant. Newton's second law thus can be written as
\[ F = \frac{dp}{dt} \] (8.20)

For a system of n particles, each with its own mass and velocity, the linear momentum \( P \) of the system as a whole is the vector sum of the linear momenta of each particle individually, that is

\[
P = p_1 + p_2 + \ldots + p_n
\]

\[
= m_1v_1 + m_2v_2 + \ldots + m_nv_n = \sum_{i=1}^{n} m_i v_i
\] (8.21)

Comparing this equation with Equation 8.13 we get

\[ P = Mv_{cm} \] (8.22)

From Equation 8.22 we define the linear momentum of a system of particles as the product of the total mass of the system and the velocity of its center of mass. Differentiating Equation 8.22 with respect to time we find

\[
\frac{dP}{dt} = M \frac{dv_{cm}}{dt} = Ma_{cm}
\] (8.23)

Comparing Equations 8.16 and 8.23 we write

\[
\sum F_{\text{ext}} = \frac{dP}{dt}
\] (8.24)

Which is the generalization of Newton's second law to a system of particles.
8.4 CONSERVATION OF LINEAR MOMENTUM

If a system is isolated, that is the resultant external force acting on the system is zero, Equation 8.24 gives

\[
\mathbf{F} = \frac{d\mathbf{P}}{dt} = 0,
\]

or

\[
\mathbf{P} = \text{constant}. \quad (8.25)
\]

This important relation is called the conservation of linear momentum principle, which states that, if no net external force acts on a system, then the total momentum of the system remains constant. Equation 8.25 can be written as

\[
\mathbf{P}_i = \mathbf{P}_f \quad (8.26)
\]

This means that, for an isolated system, the linear momentum of the system at some initial point \(i\) is equal to the linear momentum at some final point \(f\).

It is worth to mention that Equations 8.25 and 8.26 are vector equations and so both are equivalent to three separately equations corresponding the three perpendicular axes. Depending on the external force acting on the system, the linear momentum might be conserved in one or two directions, but not necessary in all directions. In another word, if only one component of the net external force acting on a system along an axis is zero, then the component of the linear momentum of the system along that axis is constant only. The other two components in this case is not constant.
Example 8.5  A cannon of mass 2000 kg rests on a smooth, horizontal surface as shown in Figure 8.6. The cannon fires, horizontally, a ball of mass 25 kg with a speed of 75 m/s relative to the earth. What is the velocity of the cannon just after it fires the ball?

Solution  We take our system to consists of the cannon and the cannonball. The two external forces, the force of gravity and the normal force, are both perpendicular to the motion of the system. Therefore, the $x$-component of the linear momentum of the system is conserved. Before firing the linear momentum of the system $P_i$ is zero, while the linear momentum of the system just after firing $P_f$ is

$$P_f = m_c v_c + m_b v_b$$

Where $c$ and $b$ refer, respectively, to the cannon and the cannonball. Conservation of linear momentum in the horizontal direction requires that

$$m_c v_c + m_b v_b = 0$$

Solving for $v_c$ yields

$$v_c = -\frac{m_b}{m_c} v_b = -\left(\frac{25}{2000}\right) 75 = -0.94 \text{ m/s}$$

The negative sign indicates that the cannon recoils to the right, in the direction opposite to the motion of the ball.
Example 8.6 Two blocks of masses \( m_1 = 1 \text{ kg} \) and \( m_2 = 2 \text{ kg} \) are connected by a spring of force constant \( k=200 \text{ N/m} \). The two blocks are free to slide along a frictionless horizontal surface, as shown in Figure 8.7. The blocks are pushed in opposite direction compressing the spring a distance of 12 cm, and then released from rest. Find the velocities of the two blocks when the spring returns to its equilibrium state.

Solution Our system is the two blocks and the spring. Therefore, the total momentum in the horizontal direction is conserved. Knowing that the system is initially at rest we obtain

\[
0 = m_1v_1 + m_2v_2
\]

So we get

\[
v_2 = -\frac{m_1}{m_2}v_1
\]

Which gives the relation between the two velocities at any instant of the motion.

Now applying the conservation of mechanical energy principle we can write

\[
\frac{1}{2}kx^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2
\]

Substituting for \( v_2 \) from the previous equation we get

\[
m_1v_1^2 + m_2\left(\frac{m_1}{m_2}\right)^2v_1^2 = kx^2
\]
Solving for $v_1$ we obtain

$$v_1 = \sqrt{\frac{km_2x^2}{m_1m_2 + m_1^2}} = \sqrt{\frac{(200)(2)(0.12)}{2 + 1}} = 4.0 \text{ m/s}$$

Again using the relation between the two velocities we get

$$v_2 = -\frac{m_1}{m_2}v_1 = -\frac{1}{2}(4.0) = -2.0 \text{ m/s}$$
PROBLEMS

8.1 Two particles are located along the x-axis. The first particle with mass $m_1 = 4$ kg has the coordinates $(2m,0,0)$, while the other particle with mass $m_2 = 8$ kg has the coordinates $(4m,0,0)$. Find the coordinates of the center of mass of the system.

8.2 A particle of mass 2 kg is located on $x = -2$ m, and a particle of mass 3 kg is on $x = 4$ m. Find the position of the center of mass.

8.3 Three masses are located as shown in Figure 8.8. Find the center of mass of the system.

8.4 Three thin rods each of length $L$ is arranged as shown in Figure 8.9. The two vertical rods have equal mass $M$, and the horizontal rod has a mass $2M$. Find the center of mass of the system.
8.5 Show that the center of mass for a rectangular plate of sides $a$ and $b$ is at the center of the plate.

8.6 A lamina in the shape of a right triangle, with dimensions $a$ and $b$ as shown in Figure 8.25, has a uniform mass per unit area. Find the coordinates of the center of mass of the lamina.

8.7 A square piece of side 10 cm is cut out of a square plate of side 30 cm. Find the coordinates of the center of mass of the plate.

8.8 A 6 kg particle moves along the x-axis with a speed of 4 m/s. A another particle of mass 4 kg moves along the x-axis with a speed of 8 m/s. Calculate the velocity of the center of mass.

8.9 Two boys one of mass 45 kg and the other of mass 35 kg, stand on a frictionless, horizontal surface holding a rigid rope of length 15 m. Starting from the ends of the rope, the boys pull themselves along the rope until they meet. How far will each boy move?

8.10 Two particles are initially at rest and 1.5 m apart. The two particles attract each other with a constant force of 2 mN,
which is the only force acting on the system. If the masses of the particles are 1 kg and 0.5 kg, find
a) the speed of the center of mass,
b) the distance from the first particle's position at which they collide.

8.11 At the instant the traffic light turns green, a car with mass 1500 kg starts from rest with constant acceleration of 2 m/s$^2$. At the same instant a truck of mass 3000 kg, traveling with constant speed of 10 m/s overtakes and passes the car.

a) How far beyond the traffic light is the center of mass of the car-truck system at $t=12$ s?
b) What is the speed of the center of mass at that instant?

8.12 Consider a body of mass 2-kg and velocity of (2i-3j) m/s.

a) Find the $x$ and $y$ components of momentum.
b) The magnitude of its total momentum.

8.13 Calculate the magnitude of the linear momentum for the following cases:

a) a proton of mass $1.67 \times 10^{-27}$ kg moving with a speed of $5 \times 10^6$ m/s,
b) a 15-g bullet moving with a speed of 500 m/s,
c) a 75-kg man running at a speed of 10 m/s, and
d) the earth of mass $5.98 \times 10^{26}$ kg, moving with an orbital speed of $2.98 \times 10^4$ m/s.

8.14 Two blocks of masses $m_1 = 3$ kg and $m_2 = 6$ kg are connected by a spring of force constant $k=800$ N/m. The two blocks are free to slide along a frictionless horizontal surface, as shown in Figure 8.12. The blocks are pulled in opposite direction stretching the spring a distance of 10 cm, and then released from rest.
a) Find the velocities of the two blocks when the spring is at its equilibrium state.
b) What is the maximum compressing distance of the spring?

8.15 A block of mass $m_1 = 2$ kg moves with speed of $v_1 = 10$ m/s to the right on a frictionless surface collides with a mass $m_2 = 8$ kg moving with velocity of $v_2 = 3$ m/s to the right. A light spring of spring constant $k = 1000$ N/m is connected to the block of mass $m_2$, as in Figure 8.13. When the blocks collide, what is the maximum compression of the spring?

8.16 A 60-kg student is standing on a cart of mass 120 kg. The cart, originally at rest, is free to slide on a smooth, horizontal surface. The student begins to walk along the cart at a constant velocity of 0.8 m/s relative to the cart.

a) What is the student's velocity relative to the ground?
b) What is the velocity of the cart relative to the ground?

8.17 A block of mass 6 kg sliding on a frictionless surface explodes into two equal pieces. One piece goes south at 4 m/s, and the other piece goes 30° north of west at 5 m/s. What was the original velocity of the block?

8.18 A block of mass 10-kg initially at rest explodes into three pieces. A 4.5-kg piece goes north at 20 m/s, and a 2-kg piece moves eastward at 60 m/s.

a) Determine the magnitude and direction of the velocity of the third piece.
b) Find the energy of the explosion.

8.19 A cart of mass 40 kg is traveling at a speed of 3 m/s along a horizontal smooth surface. A 90-kg man initially riding on the cart jumps off with zero horizontal speed. What is the final speed of the cart?

8.20 A cannon car of mass 5000 kg that rests on a smooth, horizontal surface is attached to a post by a spring of force constant of $4 \times 10^4$ N/m, as shown in Figure 8.14. The cannon fires a ball of mass 150 kg with a speed of 100 m/s, directed 30° above the horizontal.

a) What is the velocity of the cannon just after it fires the ball?

b) Find the maximum extension of the spring.

---

**Figure 8.14** Example 8.20.