Chapter 6

Section 6.2

6.2.1 (a) Reject \( H_0 \) if \( \frac{\bar{y} - 120}{18/\sqrt{25}} \leq -1.41; z = -1.61; \) reject \( H_0 \).

(b) Reject \( H_0 \) if \( \frac{\bar{y} - 42.9}{3.2/\sqrt{16}} \) is either 1) \( \leq -2.58 \) or 2) \( \geq 2.58; z = 2.75; \) reject \( H_0 \).

(c) Reject \( H_0 \) if \( \frac{\bar{y} - 14.2}{4.1/\sqrt{9}} \geq 1.13; z = 1.17; \) reject \( H_0 \).

6.2.3 (a) No, because the observed \( z \) could fall between the 0.05 and 0.01 cutoffs.

(b) Yes. If the observed \( z \) exceeded the 0.01 cutoff, it would necessarily exceed the 0.05 cutoff.

6.2.5 No, because two-sided cutoffs (for a given \( \alpha \)) are further away from 0 than one-sided cutoffs.

6.2.7 (a) \( H_0 \) should be rejected if \( \frac{\bar{y} - 12.6}{0.4/\sqrt{30}} \) is either 1) \( \leq -1.96 \) or 2) \( \geq 1.96 \). But \( \bar{y} = 12.76 \) and \( z = 2.19 \), suggesting that the machine should be readjusted.

(b) The test assumes that the \( y_i \)'s constitute a random sample from a normal distribution. Graphed, a histogram of the 30 \( y_i \)'s shows a mostly bell-shaped pattern. There is no reason to suspect that the normality assumption is not being met.

6.2.9 \( P \)-value = \( P(Z \leq -0.92) + P(Z \geq 0.92) = 0.3576 \); \( H_0 \) would be rejected if \( \alpha \) had been set at any value greater than or equal to 0.3576.

6.2.11 \( H_0 \) should be rejected if \( \frac{\bar{y} - 145.75}{9.50/\sqrt{25}} \) is either 1) \( \leq -1.96 \) or 2) \( \geq 1.96 \). Here, \( \bar{y} = 149.75 \) and \( z = 2.10 \), so the difference between $145.75 and $149.75 is statistically significant.

Section 6.3

6.3.1 (a) Given that the technique worked \( k = 24 \) times during the \( n = 52 \) occasions it was tried, \( z = \frac{24 - 52(0.40)}{\sqrt{52}(0.40)(0.60)} = 0.91 \). The latter is not larger than \( z_{.05} = 1.64 \), so \( H_0 \): \( p = 0.40 \) would not be rejected at the \( \alpha = 0.05 \) level. These data do not provide convincing evidence that transmitting predator sounds helps to reduce the number of whales in fishing waters.

(b) \( P \)-value = \( P(Z \geq 0.91) = 0.1814 \); \( H_0 \) would be rejected for any \( \alpha \geq 0.1814 \).
6.3.3 Let \( p = P(\text{current supporter is male}) \). Test \( H_0: p = 0.65 \) versus \( H_1: p < 0.65 \). Since \( n = 120 \) and \( k = \text{number of male supporters} = 72, z = \frac{72 - 120(0.65)}{\sqrt{120(0.65)(0.35)}} = -1.15 \), which is not less than or equal to \(-z_{0.05} = -1.64\), so \( H_0: p = 0.65 \) would not be rejected.

6.3.5 Let \( p = P(Y \leq 0.69315) \). Test \( H_0: p = \frac{1}{2} \) versus \( H_1: p \neq \frac{1}{2} \). Given that \( k = 26 \) and \( n = 60 \),
\[ P\text{-value} = P(X \leq 26) + P(X \geq 34) = 0.3030. \]

6.3.7 Reject \( H_0 \) if \( k \geq 4 \) gives \( \alpha = 0.50 \); reject \( H_0 \) if \( k \geq 5 \) gives \( \alpha = 0.23 \); reject \( H_0 \) if \( k \geq 6 \) gives \( \alpha = 0.06 \); reject \( H_0 \) if \( k \geq 7 \) gives \( \alpha = 0.01 \).

6.3.9 (a) \[ \alpha = P(\text{reject } H_0 \mid H_0 \text{ is true}) = P(X \leq 3 \mid p = 0.75) = \sum_{k=0}^{3} \binom{7}{k}(0.75)^k(0.25)^{7-k} = 0.07 \]

(b) \[
\begin{array}{cc}
0.75 & 0.07 \\
0.65 & 0.20 \\
0.55 & 0.39 \\
0.45 & 0.61 \\
0.35 & 0.80 \\
0.25 & 0.93 \\
0.15 & 0.99 \\
\end{array}
\]

Section 6.4

6.4.1 (a) As described in Example 6.2.1, \( H_0: \mu = 494 \) is to be tested against \( H_1: \mu \neq 494 \) using \( \pm 1.96 \) as the \( \alpha = 0.05 \) cutoffs. That is, \( H_0 \) is rejected if \( \frac{\bar{Y} - 494}{124/\sqrt{86}} \leq -1.96 \) or if \( \frac{\bar{Y} - 494}{124/\sqrt{86}} \geq 1.96 \). Equivalently, the null hypothesis is rejected if \( \bar{Y} \leq 467.8 \) or if \( \bar{Y} \geq 520.2 \). Therefore, \( 1 - \beta = P(\text{reject } H_0 \mid \mu = 500) = P(\bar{Y} \leq 467.8 \mid \mu = 500) + P(\bar{Y} \geq 520.2 \mid \mu = 500) = \)
\[
P\left( Z \leq \frac{467.8 - 500}{124/\sqrt{86}} \right) + P\left( Z \geq \frac{520.2 - 500}{124/\sqrt{86}} \right) = P(Z \leq -2.41) + P(Z \geq 1.51) =
0.0080 + 0.0655 = 0.0735.
\]

6.4.3 The null hypothesis in Question 6.2.2 is rejected if \( \bar{Y} \) is either 1) \( \leq 89.0 \) or 2) \( \geq 101.0 \).
Suppose \( \mu = 90 \). Since \( \sigma = 15 \) and \( n = 22 \), \( 1 - \beta = P(\bar{Y} \leq 89.0) + P(\bar{Y} \geq 101.0) = \)
\[
P\left( Z \leq \frac{89.0 - 90}{15/\sqrt{22}} \right) + P\left( Z \geq \frac{101.0 - 90}{15/\sqrt{22}} \right) = P(Z \leq -0.31) + P(Z \geq 3.44) = 0.3783 + 0.0003 =
0.3786.\]

Chapter 6
6.4.5 \( H_0 \) should be rejected if \( z = \frac{\bar{y} - 240}{50/\sqrt{25}} \leq -2.33 \) or, equivalently, if \( \bar{y} \leq 240 - 2.33 \cdot \frac{50}{\sqrt{25}} = 216.7 \). Suppose \( \mu = 220 \). Then \( \beta = P(\text{accept } H_0 \mid H_1 \text{ is true}) = P(\bar{Y} > 216.7 \mid \mu = 220) = P\left(Z > \frac{216.7 - 220}{50/\sqrt{25}}\right) = P(Z > -0.33) = 0.6293.\)

6.4.7 For \( \alpha = 0.10, H_0: \mu = 200 \) should be rejected if \( \bar{y} \leq 200 - 1.28 \cdot \frac{15.0}{\sqrt{n}} \). Also, \( 1 - \beta = P\left(\bar{Y} \leq 200 - 1.28 \cdot \frac{15.0}{\sqrt{n}} \mid \mu = 197\right) = 0.75, \) so \( P\left(\frac{200 - 1.28 \cdot 15.0/\sqrt{n} - 197}{15.0/\sqrt{n}}\right) = 0.75. \) But \( P(Z \leq 0.67) = 0.75, \) implying that \( \frac{200 - 1.28 \cdot 15.0/\sqrt{n} - 197}{15.0/\sqrt{n}} = 0.67. \) It follows that the smallest \( n \) satisfying the conditions placed on \( \alpha \) and \( 1 - \beta \) is 95.

6.4.9 Since \( H_1 \) is one-sided, \( H_0 \) is rejected when \( \bar{y} \geq 30 + \frac{9}{\sqrt{16}}. \) Also, \( 1 - \beta = \text{power} = P\left(\bar{Y} \geq 30 + \frac{9}{\sqrt{16}} \mid \mu = 34\right) = 0.85. \) Therefore, \( 1 - \beta = P\left(Z \geq \frac{30 + \frac{9}{\sqrt{16}} - 34}{9/\sqrt{16}}\right) = 0.85. \) But \( P(Z \geq -1.04) = 0.85, \) so \( \frac{30 + \frac{9}{\sqrt{16}} - 34}{9/\sqrt{16}} = -1.04, \) implying that \( z_{\alpha} = 0.74. \) Therefore, \( \alpha = 0.23. \)

6.4.11 In this context, \( \alpha \) is the proportion of incorrect decisions made on innocent suspects—that is, \( \frac{9}{140}, \) or 0.064. Similarly, \( \beta \) is the proportion of incorrect decisions made on guilty suspects—here, \( \frac{15}{140}, \) or 0.107. A Type I error (convicting an innocent defendant) would be considered more serious than a Type II error (acquitting a guilty defendant).

6.4.13 For a uniform pdf, \( f_{\text{max}}(y) = \frac{5}{\theta^4}, \) \( 0 \leq y \leq \theta \) when \( n = 5. \) Therefore,

\[
\alpha = P(\text{reject } H_0 \mid H_0 \text{ is true}) = P(Y_{\text{max}} \geq k \mid \theta = 2) = \int_{2}^{7} \frac{5}{y^4} \frac{1}{25} dy = 1 - \frac{k^5}{32}. \]

For \( \alpha \) to be 0.05, \( k = 1.98. \)

6.4.15 \( \beta = P(\text{accept } H_0 \mid H_1 \text{ is true}) = P(X \leq n - 1 \mid p) = 1 - P(X = n \mid p) = 1 - \left(\frac{n}{n}\right) p^n (1 - p)^0 = 1 - p^n. \) When \( \beta = 0.05, p = \sqrt[3]{0.95}. \)
6.4.17 \( 1 - \beta = P(\text{reject } H_0 \mid H_1 \text{ is true}) = P \left( Y \leq \frac{1}{2} \mid \theta \right) = \int_0^{\theta/2} (1 + \theta) y^{\theta-1} dy = \left( \frac{1}{2} \right)^{\theta+1} \frac{1}{\theta} \frac{1}{2} \frac{1}{8} = 7.18 \).

6.4.19 \( P(\text{Type II error}) = \beta = P(\text{accept } H_0 \mid H_1 \text{ is true}) = P \left( X \leq \frac{3}{2} \mid \theta = 2 \right) = \sum_{k=1}^{3} \frac{1 - \frac{1}{2}}{2} \left( \frac{1}{2} \right)^k \frac{1}{8} = 0.04 \).

6.4.21 \( \alpha = P(\text{reject } H_0 \mid H_0 \text{ is true}) = P(Y_1 + Y_2 \leq k \mid \theta = 2). \) When \( H_0 \) is true, \( Y_1 \) and \( Y_2 \) are uniformly distributed over the square defined by \( 0 \leq Y_1 \leq 2 \) and \( 0 \leq Y_2 \leq 2 \), so the joint pdf of \( Y_1 \) and \( Y_2 \) is a plane parallel to the \( Y_1Y_2 \)-axis at height \( \frac{1}{4} \left( f_{Y_1}(y_1) \cdot f_{Y_2}(y_2) = \frac{1}{2} \cdot \frac{1}{2} \right) \). By geometry, \( \alpha \) is the volume of the triangular wedge in the lower left-hand corner of the square over which \( Y_1 \) and \( Y_2 \) are defined. The hypotenuse of the triangle in the \( Y_1Y_2 \)-plane has the equation \( y_1 + y_2 = k \). Therefore, \( \alpha = \text{area of triangle} \times \text{height of wedge} = \frac{1}{2} \cdot k \cdot \frac{1}{4} = k^2/8 \). For \( \alpha \) to be 0.05, \( k = \sqrt{0.04} = 0.63 \).

Section 6.5

6.5.1 \( L(\hat{\omega}) = \prod_{i=1}^{n} (1 - p_0) y_i^{-1} p_0 = p_0^{\sum_{i=1}^{n} k_i - n} = p_0^{(1 - p_0)^{k - n}}, \) where \( k = \sum_{i=1}^{n} k_i \). From the comment following Example 5.2.1, the maximum likelihood estimate for \( p \) is \( p = \frac{n}{k} \).

Therefore, \( L(\hat{\omega}) = \left( \frac{n}{k} \right)^{\sum_{i=1}^{n} k_i - n} \), and the generalized likelihood ratio for testing \( H_0: p = p_0 \) versus \( H_1: p \neq p_0 \) is the quotient \( L(\hat{\omega}) / L(\hat{\omega}) \).

6.5.3 \( L(\hat{\omega}) = \prod_{i=1}^{n} (1/\sqrt{2\pi}) e^{-\frac{1}{2}(\gamma_i - \mu_0)^2} = (2\pi)^{-n/2} e^{\frac{1}{2} \sum_{i=1}^{n} (\gamma_i - \mu_0)^2}. \) Since \( \bar{y} \) is the maximum likelihood estimate for \( \mu \) (recall the first derivative taken in Example 5.2.4),\( L(\bar{\omega}) = (2\pi)^{-n/2} e^{\frac{1}{2} \sum_{i=1}^{n} (\gamma_i - \gamma)^2}. \) Here the generalized likelihood ratio reduces to \( \lambda = L(\bar{\omega}) / L(\hat{\omega}) = e^{-\frac{1}{2} (\bar{\gamma} - \mu_0)^2 (1/\sqrt{n})^2}. \) The null hypothesis should be rejected if \( e^{-\frac{1}{2} (\bar{\gamma} - \mu_0)^2 (1/\sqrt{n})^2} \leq \lambda^* \) or, equivalently, if \( |(\bar{\gamma} - \mu_0) / (1/\sqrt{n}) > \lambda^{**} \), where values for \( \lambda^{**} \) come from the standard normal pdf, \( f(x) \).
6.5.5 (a) \[ \lambda = \left( \frac{1}{2} \right)^n \text{e}^{[\{x/n\}^n (1 - x/n)^{n-x}] = 2^n x^n (n-x)^{n-n}.} \] Rejecting \( H_0 \) when \( 0 < \lambda \leq \lambda^* \) is equivalent to rejecting \( H_0 \) when \( x \ln x + (n-x) \ln(n-x) \geq \lambda^* \).

(b) By inspection, \( x \ln x + (n-x) \ln(n-x) \) is symmetric in \( x \). Therefore, the left-tail and right-tail critical regions will be equidistant from \( p = \frac{1}{2} \), which implies that \( H_0 \) should be rejected if \( \left| x - \frac{1}{2} \right| \geq k \), where \( k \) is a function of \( \alpha \).