Chapter 3

DISCRETE PROBABILITY DISTRIBUTIONS

- 3.1 The Binomial Distribution
- 3.2 The Geometric Distribution
- 3.3 The Hypergeometric Distribution
- 3.4 The Poisson Distribution

Section 3.1

The Binomial Distribution
Before going into the binomial distribution, we will talk about the *Bernoulli Random Variables*.

**What is a Bernoulli Random Variable?**

- A Bernoulli RV is a discrete RV that can take just two values. It can be used to model:
  - The outcome of a coin toss
  - Whether a valve is open or shut
  - Whether an item is defective or not

- Typically, the outcomes are labeled as 0 and 1
  - \( P(X=1) = p \rightarrow \) this is the probability of recording 1 or \( p \)(success)
  - \( P(X=0) = 1 - p \).

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### 3.1 The Binomial Distribution

If \( X \) is a Bernoulli Random Variable, can you determine its expectation and variance?

- \( E(X) = ? \)

- \( \text{Var}(X) = ? \)

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An experiment that has only two outcomes is often referred to as a *Bernoulli trial*. 
Consider the following random experiments and random variables:

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Random Variable of Interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flip a coin 10 times (10 Bernoulli trials)</td>
<td>X = number of heads obtained</td>
</tr>
<tr>
<td>Next 20 births at a hospital (20 Bernoulli trials)</td>
<td>X = number of male births</td>
</tr>
<tr>
<td>A machine produces 25 parts in one day (25 Bernoulli trials)</td>
<td>X = number of defective parts in a day</td>
</tr>
</tbody>
</table>

The random variables in the second column are called binomial random variables. It is the number of successes obtained within a fixed number of independent “n” trials. Success is defined according to your interest.

If \( n \) independent Bernoulli trials \( X_1, X_2, \ldots, X_n \) are performed; each with a probability “\( p \)” of recording a 1, then the random variable \( X = X_1 + X_2 + \ldots + X_n \) is said to have a binomial distribution with parameters \( n \) & \( p \)

\[ X \sim B(n, p) \]

This random variable takes the values 0, 1, 2, \ldots, \( n \)

- i.e. it counts the number of successes in the “\( n \)” Bernoulli trials. In other words: it counts the number of the Bernoulli random variables that take the value 1.

The pmf of a \( B(n, p) \) random variable is:

\[ P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \quad \text{for } x = 0, 1, 2, \ldots, n \]

\[ E(X) = np \quad \text{and} \quad \text{Var}(X) = np(1 - p) \]
**Example**

Consider the case of 4 trials. Each trial has two outcomes; 0 and 1. What is the probability that this binomial random variable take a value of 3?

If $X$ is a random variable with a binomial distribution of parameters $n$ and $p$; i.e. $X \sim B(n, p)$,

Evaluate and describe the meaning of:

$P(X = 0), \text{ and } P(X = n)$

▸ $P(X = 0) = ?$

▸ $P(X = n) = ?$
Example

If the probability is 0.05 that a certain type of columns will fail under a given axial load. What is the probability that among 16 such columns:

a) *Four columns will fail*

b) *At most 2 will fail*

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Example

Each sample of water has a 10% chance of containing a particular organic pollutant. Assume that the samples are independent with regard to the presence of the pollutant. Compute the following probabilities:

a) The probability that in the next 18 samples, exactly two contain the pollutant

b) The probability that at least four samples contain the pollutant

c) \( P(3 \leq X \leq 7) \)
Example

For the binomial random variable: $X \sim B(6, 0.7)$

a) Graph the pmf of $X$
b) Compute the cdf of $X$
c) Compute $E(X)$ and $Var(X)$
3.1 The Binomial Distribution

Example
A person enjoys playing basketball. He makes about 50% of the field goals he attempts during a game. Draw the distribution of the number of successful attempts and find the mean and variance of the number of successful attempts.

This is a binomial distribution with \( n = 6 \) and \( p = 0.5 \)

\[ X \sim B(6, 0.5) \]
**Note**
In the previous example, the distribution was symmetric about the mean value ($\mu$). (why?)

**Conclusion**: Whenever $p = 0.5$ (i.e. $B(n, 0.5)$), the graph of the binomial distribution will be symmetric regardless of the number of trials we have.

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**Example**
A fair coin is tossed eight times. Draw the distribution and compute the expectation and the variance of the number of heads obtained in the eight tosses.

This is a binomial distribution with $n = 8$ and $p = 0.5$

$$\rightarrow X \sim B(8, 0.5)$$
Example
A fair die is rolled eight times. A player wins $1 whenever a 5 or a 6 is scored, while he wins nothing if any other number is scored.

a) Draw the distribution and compute the expectation and the variance of the amount of money a player wins after rolling the die eight times.

b) What is the probability that no more than $2 is won?

This is a binomial distribution with $n = 8$ and $p = 1/3$

$\rightarrow X \sim B(8, 1/3)$
**Note**
In the previous example, the random variable $X$ counts the number of successes in $n$ Bernoulli trials. The probability of success was $p$.

What about the random variable $Y = n - X$?
$Y$ is a random variable that counts the number of failures in $n$ Bernoulli trials, where the failure probability is $1 - p$.

$$Y \sim B(n, 1 - p)$$

**In the previous example:** $Y \sim B(8, 2/3)$
The graph of $B(8, 2/3)$ is a reflection of $B(8, 1/3)$

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**Example (Milk container contents)** p 163
Recall that there is a probability of 0.261 that a milk container is underweight. Suppose that the milk containers are shipped to retail outlets in boxes of 20 containers. What is the distribution of the number of underweight containers in a box?

Each individual milk container represents a Bernoulli trial.

$X$ is a random variable representing the number of underweight containers in a box, which is also a Binomial random variable with parameters $n = 20$ and $p = 0.261$.

$$\rightarrow X \sim B(20, 0.261)$$
Example \((Air \, Force \, planes) \, p \, 166\)

A group of 16 planes should always be ready for immediate launch. There is a probability of 0.25 that the engines of a particular plane will not start at a given attempt.

a) Find the expectation, variance, and the distribution of the number of planes successfully launched?

b) Find the probability that exactly 12 planes are successfully launched.
Cont... *air force example*

**Example** *(Tossing a coin n times) p 167*

A fair coin is tossed $n$ times. What are the expectation and the variance of the number of heads obtained?
Suppose that $X \sim B(n, p)$
- This means that $X$ is a random variable that counts the number of successes in $n$ independent Bernoulli trials.

The random variable $Y = X/n$ is the proportion of success
- $E(Y) = p$, $\text{Var}(Y) = p(1-p)/n$

For example: When a fair coin is tossed $n$ times
- $X$: RV representing the number of heads obtained
- $Y$: RV representing the proportion of heads obtained = $X/n$
  - $E(Y) = p = 0.5$, $\text{Var}(Y) = p(1-p)/n = (0.5)(0.5)/n = 1/(4n)$
  - $\text{OR}$ $\text{Var}(Y) = \text{Var}(X)/n^2 = 1/(4n)$ (since $a = 1/n^2$)

The variance of the proportion $Y$ decreases as the number of trials $n$ increases. Table 3.10 in your textbook shows that the proportion of heads approaches the mean value of 0.5 as the number of tosses increases.