CHAPTER OUTLINE

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2-1 Sample Spaces and Events

2-1.1 Random Experiments

Definition

An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a random experiment.
2-1 Sample Spaces and Events

2-1.2 Sample Spaces

**Definition**

The set of all possible outcomes of a random experiment is called the *sample space* of the experiment. The sample space is denoted as $S$. 
2-1 Sample Spaces and Events

2-1.2 Sample Spaces

Example 2-1

Consider an experiment in which you select a molded plastic part, such as a connector, and measure its thickness. The possible values for thickness depend on the resolution of the measuring instrument, and they also depend on upper and lower bounds for thickness. However, it might be convenient to define the sample space as simply the positive real line

\[ S = R^+ = \{x \mid x > 0\} \]

because a negative value for thickness cannot occur.
Example 2-1 (continued)

If it is known that all connectors will be between 10 and 11 millimeters thick, the sample space could be

\[ S = \{ x | 10 < x < 11 \} \]

If the objective of the analysis is to consider only whether a particular part is low, medium, or high for thickness, the sample space might be taken to be the set of three outcomes:

\[ S = \{ \text{low, medium, high} \} \]

If the objective of the analysis is to consider only whether or not a particular part conforms to the manufacturing specifications, the sample space might be simplified to the set of two outcomes

\[ S = \{ \text{yes, no} \} \]

that indicate whether or not the part conforms.
Example 2-2

If two connectors are selected and measured, the extension of the positive real line \( R \) is to take the sample space to be the positive quadrant of the plane:

\[
S = R^+ \times R^+
\]

If the objective of the analysis is to consider only whether or not the parts conform to the manufacturing specifications, either part may or may not conform. We abbreviate \( yes \) and \( no \) as \( y \) and \( n \). If the ordered pair \( yn \) indicates that the first connector conforms and the second does not, the sample space can be represented by the four outcomes:

\[
S = \{yy, yn, ny, nn\}
\]
Example 2-2 (continued)

As another example, consider an experiment in which the thickness is measured until a connector fails to meet the specifications. The sample space can be represented as

\[ S = \{ n, yn, yyn, yyyy, yyyy, and \text{ so forth} \} \]
2-1 Sample Spaces and Events

Tree Diagrams

• Sample spaces can also be described graphically with **tree diagrams**.
  – When a sample space can be constructed in several steps or stages, we can represent each of the $n_1$ ways of completing the first step as a branch of a tree.
  – Each of the ways of completing the second step can be represented as $n_2$ branches starting from the ends of the original branches, and so forth.
Example 2-4

• An automobile manufacturer provides vehicles equipped with selected options. Each vehicle is ordered
  – With or without an automatic transmission
  – With or without air-conditioning
  – With one of three choices of a stereo system
  – With one of four exterior colors

• If the sample space consists of the set of all possible vehicle types, what is the number of outcomes in the sample space? The sample space contains 48 outcomes. The tree diagram for the different types of vehicles is displayed in Fig. 2-6.
Figure 2-6 Tree diagram for different types of vehicles.
2-1 Sample Spaces and Events

2-1.3 Events

**Definition**

An event is a subset of the sample space of a random experiment.
2-1 Sample Spaces and Events

2-1.3 Events

**Basic Set Operations**

- The **union** of two events is the event that consists of all outcomes that are contained in either of the two events. We denote the union as $E_1 \cup E_2$.
- The **intersection** of two events is the event that consists of all outcomes that are contained in both of the two events. We denote the intersection as $E_1 \cap E_2$.
- The **complement** of an event in a sample space is the set of outcomes in the sample space that are not in the event. We denote the component of the event $E$ as $E'$.
2-1 Sample Spaces and Events

2-1.3 Events

Example 2-6

Consider the sample space $S = \{yy, yn, ny, nn\}$ in Example 2-2. Suppose that the set of all outcomes for which at least one part conforms is denoted as $E_1$. Then,

$$E_1 = \{yy, yn, ny\}$$

The event in which both parts do not conform, denoted as $E_2$, contains only the single outcome, $E_2 = \{nn\}$. Other examples of events are $E_3 = \emptyset$, the null set, and $E_4 = S$, the sample space. If $E_5 = \{yn, ny, nn\}$,

$$E_1 \cup E_5 = S \quad E_1 \cap E_5 = \{yn, ny\} \quad E'_1 = \{nn\}$$
Definition

Two events, denoted as $E_1$ and $E_2$, such that

$$E_1 \cap E_2 = \emptyset$$

are said to be mutually exclusive.
2-1 Sample Spaces and Events

Venn Diagrams

Sample space $S$ with events $A$ and $B$

$(A \cup B) \cap C$

$(A \cap C)'$

Figure 2-8 Venn diagrams.
2-2 Interpretations of Probability

2-2.1 Introduction

Probability

- Used to quantify likelihood or chance
- Used to represent risk or uncertainty in engineering applications
- Can be interpreted as our degree of belief or relative frequency
Possible Values for Probabilities
2-2 Interpretations of Probability

Equally Likely Outcomes

Whenever a sample space consists of $N$ possible outcomes that are equally likely, the probability of each outcome is $1/N$. 
Example 2-15

Assume that 30% of the laser diodes in a batch of 100 meet the minimum power requirements of a specific customer. If a laser diode is selected randomly, that is, each laser diode is equally likely to be selected, our intuitive feeling is that the probability of meeting the customer’s requirements is 0.30.

Let $E$ denote the subset of 30 diodes that meet the customer's requirements. Because $E$ contains 30 outcomes and each outcome has probability 0.01, we conclude that the probability of $E$ is 0.3. The conclusion matches our intuition. Figure 2-10 illustrates this example.
2-2 Interpretations of Probability

Figure 2-11 Probability of the event $E$ is the sum of the probabilities of the outcomes in $E$
2-2 Interpretations of Probability

Definition

For a discrete sample space, the probability of an event $E$, denoted as $P(E)$, equals the sum of the probabilities of the outcomes in $E$. 
2-2 Interpretations of Probability

Example 2-16

A random experiment can result in one of the outcomes \{a, b, c, d\} with probabilities 0.1, 0.3, 0.5, and 0.1, respectively. Let \(A\) denote the event \{a, b\}, \(B\) the event \{b, c, d\}, and \(C\) the event \{d\}. Then,

\[
P(A) = 0.1 + 0.3 = 0.4
\]
\[
P(B) = 0.3 + 0.5 + 0.1 = 0.9
\]
\[
P(C) = 0.1
\]

Also, \(P(A') = 0.6, P(B') = 0.1, \) and \(P(C') = 0.9. \) Furthermore, because \(A \cap B = \{b\}, P(A \cap B) = 0.3.\) Because \(A \cup B = \{a, b, c, d\}, P(A \cup B) = 0.1 + 0.3 + 0.5 + 0.1 = 1.\) Because \(A \cap C\) is the null set, \(P(A \cap C) = 0.\)
## 2-2 Interpretations of Probability

### 2-2.2 Axioms of Probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

If $S$ is the sample space and $E$ is any event in a random experiment,

1. $P(S) = 1$
2. $0 \leq P(E) \leq 1$
3. For two events $E_1$ and $E_2$ with $E_1 \cap E_2 = \emptyset$

\[
P(E_1 \cup E_2) = P(E_1) + P(E_2)
\]
2-3 Addition Rules

Probability of a Union

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{2-5}\]
2-3 Addition Rules

Mutually Exclusive Events

If \( A \) and \( B \) are mutually exclusive events,

\[
P(A \cup B) = P(A) + P(B)
\]  
(2-6)
2-3 Addition Rules

Three Events

\[ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) \]
\[ - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \]  \hspace{1cm} (2-7)
A collection of events, $E_1, E_2, \ldots, E_k$, is said to be mutually exclusive if for all pairs,

$$E_i \cap E_j = \emptyset$$

For a collection of mutually exclusive events,

$$P(E_1 \cup E_2 \cup \ldots \cup E_k) = P(E_1) + P(E_2) + \ldots + P(E_k) \quad (2-8)$$
2-3 Addition Rules

Figure 2-12 Venn diagram of four mutually exclusive events
Example 2-21

A simple example of mutually exclusive events will be used quite frequently. Let $X$ denote the pH of a sample. Consider the event that $X$ is greater than 6.5 but less than or equal to 7.8. This probability is the sum of any collection of mutually exclusive events with union equal to the same range for $X$. One example is

$$P(6.5 < X \leq 7.8) = P(6.5 < X \leq 7.0) + P(7.0 < X \leq 7.5) + P(7.5 < X \leq 7.8)$$

Another example is

$$P(6.5 < X \leq 7.8) = P(6.5 < X \leq 6.6) + P(6.6 < X \leq 7.1)
+ P(7.1 < X \leq 7.4) + P(7.4 < X \leq 7.8)$$

The best choice depends on the particular probabilities available.
To introduce conditional probability, consider an example involving manufactured parts.

Let $D$ denote the event that a part is defective and let $F$ denote the event that a part has a surface flaw.

Then, we denote the probability of $D$ given, or assuming, that a part has a surface flaw as $P(D|F)$. This notation is read as the **conditional probability** of $D$ given $F$, and it is interpreted as the probability that a part is defective, given that the part has a surface flaw.
2-4 Conditional Probability

Figure 2-13 Conditional probabilities for parts with surface flaws
The conditional probability of an event $B$ given an event $A$, denoted as $P(B|A)$, is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

for $P(A) > 0$. 

(2-9)
### Table 2-3 Parts Classified

<table>
<thead>
<tr>
<th></th>
<th>Surface Flaws</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Yes (event $D$)</td>
<td>No</td>
<td>Total</td>
</tr>
<tr>
<td>Defective</td>
<td>10</td>
<td>18</td>
<td>38</td>
</tr>
<tr>
<td>No</td>
<td>30</td>
<td>342</td>
<td>362</td>
</tr>
<tr>
<td>Total</td>
<td>40</td>
<td>360</td>
<td>400</td>
</tr>
</tbody>
</table>

**EXAMPLE 2-16**

Table 2-3 provides an example of 400 parts classified by surface flaws and as (functionally) defective. For this table the conditional probabilities match those discussed previously in this section. For example, of the parts with surface flaws (40 parts) the number defective is 10. Therefore,

$$P(D|F) = \frac{10}{40} = 0.25$$

and of the parts without surface flaws (360 parts) the number defective is 18. Therefore,

$$P(D|F') = \frac{18}{360} = 0.05$$

**EXAMPLE 2-17**

Again consider the 400 parts in Table 2-3. From this table

$$P(D|F) = \frac{P(D \cap F)}{P(F)} = \frac{10}{400} \div \frac{40}{400} = \frac{10}{40}$$
### 2-4 Conditional Probability

<table>
<thead>
<tr>
<th></th>
<th>Employed (E)</th>
<th>Unemployed (E')</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>With Sugar (W)</td>
<td>20</td>
<td>40</td>
<td>60</td>
</tr>
<tr>
<td>Without Sugar (W')</td>
<td>80</td>
<td>60</td>
<td>140</td>
</tr>
<tr>
<td>total</td>
<td>100</td>
<td>100</td>
<td>200</td>
</tr>
</tbody>
</table>

\[
P(E \mid W) = \frac{20}{60}
\]
\[
P(W \mid E) = \frac{20}{100}
\]
\[
P(E \mid W') =
\]
\[
P(W \mid E') =
\]
\[
P(W' \mid E) =
\]
\[
P(E' \mid W) =
\]
2-5 Multiplication and Total Probability Rules

2-5.1 Multiplication Rule

\[ P(A \cap B) = P(B|A)P(A) = P(A|B)P(B) \]  \hspace{1cm} (2-10)
Example 2-26

The probability that the first stage of a numerically controlled machining operation for high-rpm pistons meet specifications is 0.90. Failures are due to metal variations, fixture alignment, cutting blade condition, vibration, and ambient environmental conditions. Given that the first stage meets specifications the probability that a second stage of machining meets specifications is 0.95. What is the probability that both stages meet specifications?

Let $A$ and $B$ denote the events that the first and second stages meet specifications, respectively. The probability requested is

$$P(A \cap B) = P(B|A)P(A) = 0.95(0.90) = 0.855$$

Although it is also true that $P(A \cap B) = P(A|B)P(B)$, the information provided in the problem does not match this second formulation.
2-5 Multiplication and Total Probability Rules

2-5.2 Total Probability Rule

**Figure 2-15** Partitioning an event into two mutually exclusive subsets.

**Figure 2-16** Partitioning an event into several mutually exclusive subsets.
2-5 Multiplication and Total Probability Rules

2-5.2 Total Probability Rule (two events)

For any events $A$ and $B$,

$$
P(B) = P(B \cap A) + P(B \cap A') = P(B|A)P(A) + P(B|A')P(A') \tag{2-11}$$
2-5 Multiplication and Total Probability Rules

Example 2-27

Consider the contamination discussion at the start of this section. The information is summarized here:

<table>
<thead>
<tr>
<th>Probability of Failure</th>
<th>Level of Contamination</th>
<th>Probability of Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>High</td>
<td>0.2</td>
</tr>
<tr>
<td>0.005</td>
<td>Not High</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Let $F$ denote the event that the product fails, and let $H$ denote the event that the chip is exposed to high levels of contamination. The requested probability is $P(F)$, and the information provided can be represented as:

$$P(F|H) = 0.10 \quad \text{and} \quad P(F|H') = 0.005$$
$$P(H) = 0.20 \quad \text{and} \quad P(H') = 0.80$$

From Equation 2-11,

$$P(F) = 0.10(0.20) + 0.005(0.80) = 0.024$$

which can be interpreted as just the weighted average of the two probabilities of failure.
2-5 Multiplication and Total Probability Rules

Total Probability Rule (multiple events)

Assume $E_1, E_2, \ldots, E_k$ are $k$ mutually exclusive and exhaustive sets. Then

$$P(B) = P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k)$$

$$= P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k) \quad (2-12)$$
Example 2-22

Continuing with the semiconductor manufacturing example, assume the following probabilities for product failure subject to levels of contamination in manufacturing:

<table>
<thead>
<tr>
<th>Probability of Failure</th>
<th>Level of Contamination</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>High</td>
</tr>
<tr>
<td>0.01</td>
<td>Medium</td>
</tr>
<tr>
<td>0.001</td>
<td>Low</td>
</tr>
</tbody>
</table>

In a particular production run, 20% of the chips are subjected to high levels of contamination, 30% to medium levels of contamination, and 50% to low levels of contamination. What is the probability that a product using one of these chips fails? Let

- $H$ denote the event that a chip is exposed to high levels of contamination
- $M$ denote the event that a chip is exposed to medium levels of contamination
- $L$ denote the event that a chip is exposed to low levels of contamination

Then,

$$P(F) = P(F \mid H)P(H) + P(F \mid M)P(M) + P(F \mid L)P(L)$$

$$= 0.10(0.20) + 0.01(0.30) + 0.001(0.50) = 0.0235$$
Example 2-22

\[
P(\text{Fail}) = 0.02 + 0.003 + 0.0005 = 0.0235
\]
2-6 Independence

Definition (two events)

Two events are independent if any one of the following equivalent statements is true:

1. \( P(A|B) = P(A) \)
2. \( P(B|A) = P(B) \)
3. \( P(A \cap B) = P(A)P(B) \)  

(2-13)
2-6 Independence

Definition (multiple events)

The events $E_1, E_2, \ldots, E_n$ are independent if and only if for any subset of these events $E_{i_1}, E_{i_2}, \ldots, E_{i_k}$,

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = P(E_{i_1}) \times P(E_{i_2}) \times \cdots \times P(E_{i_k})$$  \hspace{1cm} (2-14)
Example 2-26

Assume that the probability that a wafer contains a large particle of contamination is 0.01 and that the wafers are independent; that is, the probability that a wafer contains a large particle is not dependent on the characteristics of any of the other wafers. If 15 wafers are analyzed, what is the probability that no large particles are found?

Let $E_i$ denote the event that the $i$th wafer contains no large particles, $i = 1, 2, \ldots, 15$. Then, $P(E_i) = 0.99$. The probability requested can be represented as $P(E_1 \cap E_2 \cap \cdots \cap E_{15})$. From the independence assumption and Equation 2-10,

$$P(E_1 \cap E_2 \cap \cdots \cap E_{15}) = P(E_1) \times P(E_2) \times \cdots \times P(E_{15}) = 0.99^{15} = 0.86$$
Counting
In many probability problems, the big obstacle is finding the total number of outcomes, and this section presents several methods for finding such numbers without directly listing and counting the possibilities.
Fundamental Counting Rule

For a sequence of two events in which the first event can occur \( m \) ways and the second event can occur \( n \) ways, the events together can occur a total of \( m \cdot n \) ways.
**EXAMPLE** Burglary Basics The typical home alarm system has a code that consists of four digits. The digits (0 through 9) can be repeated, and they must be entered in the correct order. Assume that you plan to gain access by trying codes until you find the correct one. How many different codes are possible?

**SOLUTION** There are 10 possible values for each of the four digits, so the number of different possible codes is $10 \cdot 10 \cdot 10 \cdot 10 = 10,000$. Although all
Notation

The **factorial symbol** ! denotes the product of decreasing positive whole numbers.

For example,

\[ 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24. \]

By special definition, \(0! = 1\).
Factorial Rule

A collection of \(n\) different items can be arranged in order \(n!\) different ways. (This **factorial rule** reflects the fact that the first item may be selected in \(n\) different ways, the second item may be selected in \(n - 1\) ways, and so on.)
EXAMPLE Routes to All 50 Capitals Because of your success in a statistics course, you have been hired by the Gallup Organization, and your first assignment is to conduct a survey in each of the 50 state capitals. As you plan your route of travel, you want to determine the number of different possible routes. How many different routes are possible?

SOLUTION By applying the factorial rule, we know that 50 items can be arranged in order 50! different ways. That is, the 50 state capitals can be arranged 50! ways, so the number of different routes is 50!, or

$$30,414,093,201,713,378,043,612,608,166,064,768,$$

$$844,377,641,568,960,512,000,000,000,000,000$$

Do it for three cities (A,B, C)??
Permutations Rule
(when items are all different)

Requirements:

1. There are \( n \) different items available. (This rule does not apply if some of the items are identical to others.)

2. We select \( r \) of the \( n \) items (without replacement).

3. We consider rearrangements of the same items to be different sequences. (The permutation of \( ABC \) is different from \( CBA \) and is counted separately.)

If the preceding requirements are satisfied, the number of permutations (or sequences) of \( r \) items selected from \( n \) available items (without replacement) is

\[
n^P_r = \frac{n!}{(n - r)!}
\]
EXAMPLE Television Programming You have just been hired to determine the programming for the Fox television network. When selecting the shows to be shown on Monday night, you find that you have 27 shows available and you must select 4 of them. Because of lead-in effects, the order of the shows is important. How many different sequences of 4 shows are possible when there are 27 shows available?

SOLUTION We need to select \( r = 4 \) shows from \( n = 27 \) that are available. The number of different arrangements is found as shown:

\[
nP_r = \frac{n!}{(n - r)!} = \frac{27!}{(27 - 4)!} = 421,200
\]

There are 421,200 different possible arrangements of 4 shows selected from the 27 that are available.
Combinations Rule

Requirements:

1. There are \( n \) different items available.
2. We select \( r \) of the \( n \) items (without replacement).
3. We consider rearrangements of the same items to be the same. (The combination of ABC is the same as CBA.)

If the preceding requirements are satisfied, the number of combinations of \( r \) items selected from \( n \) different items is

\[
_{n}C_{r} = \frac{n!}{(n - r)! \cdot r!}
\]
When different orderings of the same items are to be counted separately, we have a permutation problem, but when different orderings are not to be counted separately, we have a combination problem.
EXAMPLE Elected Offices The Board of Trustees at the author’s college has 9 members. Each year, they elect a 3-person committee to oversee buildings and grounds. Each year, they also elect a chairperson, vice chairperson, and secretary.

a. When the board elects the buildings and grounds committee, how many different 3-person committees are possible?

b. When the board elects the 3 officers (chairperson, vice chairperson, and secretary), how many different slates of candidates are possible?
SOLUTION  Note that order is irrelevant when electing the buildings and grounds committee. When electing officers, however, different orders are counted separately.

a. Because order does not count for the committees, we want the number of combinations of $r = 3$ people selected from the $n = 9$ available people. We get

$$nC_r = \frac{n!}{(n - r)!r!} = \frac{9!}{(9 - 3)!3!} = \frac{362,880}{4320} = 84$$

b. Because order does count with the slates of candidates, we want the number of sequences (or permutations) of $r = 3$ people selected from the $n = 9$ available people. We get

$$nP_r = \frac{n!}{(n - r)!} = \frac{9!}{(9 - 3)!} = \frac{362,880}{720} = 504$$

There are 84 different possible committees of 3 board members, but there are 504 different possible slates of candidates.