3

Discrete Random Variables and Probability Distributions

CHAPTER OUTLINE

3-1 DISCRETE RANDOM VARIABLES
3-2 PROBABILITY DISTRIBUTIONS AND PROBABILITY MASS FUNCTIONS
3-3 CUMULATIVE DISTRIBUTION FUNCTIONS
3-4 MEAN AND VARIANCE OF A DISCRETE RANDOM VARIABLE
3-5 DISCRETE UNIFORM DISTRIBUTION
3-6 BINOMIAL DISTRIBUTION
3-7 GEOMETRIC AND NEGATIVE BINOMIAL DISTRIBUTIONS
  3-7.1 Geometric Distribution
  3-7.2 Negative Binominal Distribution
3-8 HYPERGEOMETRIC DISTRIBUTION
3-9 POISSON DISTRIBUTION
LEARNING OBJECTIVES

After careful study of this chapter you should be able to do the following:

1. Determine probabilities from probability mass functions and the reverse
2. Determine probabilities from cumulative distribution functions and cumulative distribution functions from probability mass functions, and the reverse
3. Calculate means and variances for discrete random variables
4. Understand the assumptions for each of the discrete probability distributions presented
5. Select an appropriate discrete probability distribution to calculate probabilities in specific applications
6. Calculate probabilities, determine means and variances for each of the discrete probability distributions presented
3-2 Probability Distributions and Probability Mass Functions

Definition

For a discrete random variable $X$ with possible values $x_1, x_2, \ldots, x_n$, a probability mass function is a function such that

1. $f(x_i) \geq 0$
2. $\sum_{i=1}^{n} f(x_i) = 1$
3. $f(x_i) = P(X = x_i)$ (3-1)
Cumulative Distribution Functions

Definition

The cumulative distribution function of a discrete random variable $X$, denoted as $F(x)$, is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

For a discrete random variable $X$, $F(x)$ satisfies the following properties.

(1) $F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$

(2) $0 \leq F(x) \leq 1$

(3) If $x \leq y$, then $F(x) \leq F(y)$  \hspace{1cm} (3-2)
Example 3-8

Suppose that a day’s production of 850 manufactured parts contains 50 parts that do not conform to customer requirements. Two parts are selected at random, without replacement, from the batch. Let the random variable \( X \) equal the number of nonconforming parts in the sample. What is the cumulative distribution function of \( X \)?

The question can be answered by first finding the probability mass function of \( X \).

\[
\begin{align*}
P(X = 0) &= \frac{800}{850} \cdot \frac{799}{849} = 0.886 \\
P(X = 1) &= 2 \cdot \frac{800}{850} \cdot \frac{50}{849} = 0.111 \\
P(X = 2) &= \frac{50}{850} \cdot \frac{49}{849} = 0.003
\end{align*}
\]

Therefore,

\[
\begin{align*}
F(0) &= P(X \leq 0) = 0.886 \\
F(1) &= P(X \leq 1) = 0.886 + 0.111 = 0.997 \\
F(2) &= P(X \leq 2) = 1
\end{align*}
\]

The cumulative distribution function for this example is graphed in Fig. 3-4. Note that \( F(x) \) is defined for all \( x \) from \(-\infty < x < \infty\) and not only for 0, 1, and 2.
Example 3-8

Figure 3-4 Cumulative distribution function for Example 3-8.
3-4 Mean and Variance of a Discrete Random Variable

**Definition**

The **mean** or **expected value** of the discrete random variable $X$, denoted as $\mu$ or $E(X)$, is

$$\mu = E(X) = \sum_x xf(x)$$  \hspace{1cm} (3-3)

The **variance** of $X$, denoted as $\sigma^2$ or $V(X)$, is

$$\sigma^2 = V(X) = E(X - \mu)^2 = \sum_x (x - \mu)^2 f(x) = \sum_x x^2 f(x) - \mu^2$$

The **standard deviation** of $X$ is $\sigma = \sqrt{\sigma^2}$.
Figure 3-5 A probability distribution can be viewed as a loading with the mean equal to the balance point. Parts (a) and (b) illustrate equal means, but Part (a) illustrates a larger variance.
The probability distribution illustrated in Parts (a) and (b) differ even though they have equal means and equal variances.
Example 3-11

The number of messages sent per hour over a computer network has the following distribution:

<table>
<thead>
<tr>
<th>$x$ = number of messages</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>0.08</td>
<td>0.15</td>
<td>0.30</td>
<td>0.20</td>
<td>0.20</td>
<td>0.07</td>
</tr>
</tbody>
</table>

Determine the mean and standard deviation of the number of messages sent per hour.

\[
E(X) = 10(0.08) + 11(0.15) + \cdots + 15(0.07) = 12.5
\]
\[
V(X) = 10^2(0.08) + 11^2(0.15) + \cdots + 15^2(0.07) - 12.5^2 = 1.85
\]
\[
\sigma = \sqrt{V(X)} = \sqrt{1.85} = 1.36
\]
A random variable $X$ has a **discrete uniform distribution** if each of the $n$ values in its range, say, $x_1, x_2, \ldots, x_n$, has equal probability. Then,

$$f(x_i) = \frac{1}{n}$$

(3-5)
Example 3-13

The first digit of a part’s serial number is equally likely to be any one of the digits 0 through 9. If one part is selected from a large batch and $X$ is the first digit of the serial number, $X$ has a discrete uniform distribution with probability 0.1 for each value in $R = \{0, 1, 2, \ldots, 9\}$. That is,

$$f(x) = 0.1$$

for each value in $R$. The probability mass function of $X$ is shown in Fig. 3-7.
3-5 Discrete Uniform Distribution

**Figure 3-7** Probability mass function for a discrete uniform random variable.
3-5 Discrete Uniform Distribution

Mean and Variance

Suppose $X$ is a discrete uniform random variable on the consecutive integers $a, a + 1, a + 2, \ldots, b$, for $a \leq b$. The mean of $X$ is

$$
\mu = E(X) = \frac{b + a}{2}
$$

The variance of $X$ is

$$
\sigma^2 = \frac{(b - a + 1)^2 - 1}{12}
$$

(3-6)
As in Example 3-1, let the random variable $X$ denote the number of the 48 voice lines that are in use at a particular time. Assume that $X$ is a discrete uniform random variable with a range of 0 to 48. Then,

$$E(X) = (48 + 0)/2 = 24$$

and

$$\sigma = \left\{\frac{(48 - 0 + 1)^2 - 1}{12}\right\}^{1/2} = 14.14$$
3-6 Binomial Distribution

Random experiments and random variables

1. Flip a coin 10 times. Let $X =$ number of heads obtained.
2. A worn machine tool produces 1% defective parts. Let $X =$ number of defective parts in the next 25 parts produced.
3. Each sample of air has a 10% chance of containing a particular rare molecule. Let $X =$ the number of air samples that contain the rare molecule in the next 18 samples analyzed.
4. Of all bits transmitted through a digital transmission channel, 10% are received in error. Let $X =$ the number of bits in error in the next five bits transmitted.
3-6 Binomial Distribution

Random experiments and random variables

5. A multiple choice test contains 10 questions, each with four choices, and you guess at each question. Let \( X \) = the number of questions answered correctly.

6. In the next 20 births at a hospital, let \( X \) = the number of female births.

7. Of all patients suffering a particular illness, 35% experience improvement from a particular medication. In the next 100 patients administered the medication, let \( X \) = the number of patients who experience improvement.
3-6 Binomial Distribution

**Definition**

A random experiment consists of $n$ Bernoulli trials such that

1. The trials are independent
2. Each trial results in only two possible outcomes, labeled as “success” and “failure”
3. The probability of a success in each trial, denoted as $p$, remains constant

The random variable $X$ that equals the number of trials that result in a success has a **binomial random variable** with parameters $0 < p < 1$ and $n = 1, 2, \ldots$. The probability mass function of $X$ is

$$f(x) = \binom{n}{x} p^x (1 - p)^{n-x} \quad x = 0, 1, \ldots, n$$

(3-7)
3-6 Binomial Distribution

Figure 3-8 Binomial distributions for selected values of $n$ and $p$. 
Binomial PDF's for $p=0.5$
Each sample of water has a 10% chance of containing a particular organic pollutant. Assume that the samples are independent with regard to the presence of the pollutant. Find the probability that in the next 18 samples, exactly 2 contain the pollutant.

Let $X =$ the number of samples that contain the pollutant in the next 18 samples analyzed. Then $X$ is a binomial random variable with $p = 0.1$ and $n = 18$. Therefore,

$$P(X = 2) = \binom{18}{2} (0.1)^2 (0.9)^{16}$$

Now $\binom{18}{2} = 18!/2!16! = 18(17)/2 = 153$. Therefore,

$$P(X = 2) = 153(0.1)^2(0.9)^{16} = 0.284$$
3-6 Binomial Distribution

Example 3-18

Determine the probability that at least four samples contain the pollutant. The requested probability is

$$P(X \geq 4) = \sum_{x=4}^{18} \binom{18}{x} (0.1)^x (0.9)^{18-x}$$

However, it is easier to use the complementary event,

$$P(X \geq 4) = 1 - P(X < 4) = 1 - \sum_{x=0}^{3} \binom{18}{x} (0.1)^x (0.9)^{18-x}$$

$$= 1 - [0.150 + 0.300 + 0.284 + 0.168] = 0.098$$

Determine the probability that $3 \leq X < 7$. Now

$$P(3 \leq X < 7) = \sum_{x=3}^{6} \binom{18}{x} (0.1)^x (0.9)^{18-x}$$

$$= 0.168 + 0.070 + 0.022 + 0.005$$

$$= 0.265$$
3-6 Binomial Distribution

Question

3-69. A multiple choice test contains 25 questions, each with four answers. Assume a student just guesses on each question.

(a) What is the probability that the student answers more than 20 questions correctly?
3-6 Binomial Distribution

Mean and Variance

If $X$ is a binomial random variable with parameters $p$ and $n$,

\[ \mu = E(X) = np \quad \text{and} \quad \sigma^2 = V(X) = np(1 - p) \]  

(3-8)
Example 3-19

For the number of transmitted bits received in error in Example 3-16, $n = 4$ and $p = 0.1$, so

$$E(X) = 4(0.1) = 0.4 \quad \text{and} \quad V(X) = 4(0.1)(0.9) = 0.36$$

and these results match those obtained from a direct calculation in Example 3-9.
3-7 Geometric Distribution

Example 3-20

The probability that a bit transmitted through a digital transmission channel is received in error is 0.1. Assume the transmissions are independent events, and let the random variable $X$ denote the number of bits transmitted until the first error.

Then, $P(X = 5)$ is the probability that the first four bits are transmitted correctly and the fifth bit is in error. This event can be denoted as $\{OOOOE\}$, where $O$ denotes an okay bit. Because the trials are independent and the probability of a correct transmission is 0.9,

$$P(X = 5) = P(\text{OOOOE}) = 0.9^4 \times 0.1 = 0.066$$

Note that there is some probability that $X$ will equal any integer value. Also, if the first trial is a success, $X = 1$. Therefore, the range of $X$ is $\{1, 2, 3, \ldots \}$, that is, all positive integers.
In a series of Bernoulli trials (independent trials with constant probability $p$ of a success), let the random variable $X$ denote the number of trials until the first success. Then $X$ is a geometric random variable with parameter $0 < p < 1$ and

$$f(x) = (1 - p)^{x-1}p \quad x = 1, 2, \ldots$$

(3-9)
Figure 3-9. Geometric distributions for selected values of the parameter $p$. 
The probability that a wafer contains a large particle of contamination is 0.01. If it is assumed that the wafers are independent, what is the probability that exactly 125 wafers need to be analyzed before a large particle is detected?

Let $X$ denote the number of samples analyzed until a large particle is detected. Then $X$ is a geometric random variable with $p = 0.01$. The requested probability is

$$P(X = 125) = (0.99)^{124}0.01 = 0.0029$$
3-7 Geometric Distribution

Definition

If $X$ is a geometric random variable with parameter $p$,

$$
\mu = E(X) = 1/p \quad \text{and} \quad \sigma^2 = V(X) = (1 - p)/p^2 \quad (3-10)
$$
Example 3.22

Consider the transmission of bits in Example 3-20. Here, $p = 0.1$. The mean number of transmissions until the first error is $1/0.1 = 10$. The standard deviation of the number of transmissions before the first error is

$$
\sigma = \left[ \frac{(1 - 0.1)/0.1^2} \right]^{1/2} = 9.49
$$
3-9 Poisson Distribution

Definition

Given an interval of real numbers, assume events occur at random throughout the interval. If the interval can be partitioned into subintervals of small enough length such that

1. the probability of more than one event in a subinterval is zero,
2. the probability of one event in a subinterval is the same for all subintervals and proportional to the length of the subinterval, and
3. the event in each subinterval is independent of other subintervals, the random experiment is called a Poisson process.

The random variable $X$ that equals the number of events in the interval is a Poisson random variable with parameter $0 < \lambda$, and the probability mass function of $X$ is

$$f(x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad x = 0, 1, 2, \ldots \quad (3-16)$$
3-9 Poisson Distribution

Applications:

- **Intervals:**
  - Length
  - time,
  - area,
  - volume

- **Counts:**
  1. particles of contamination in semiconductor
  2. flaws in rolls of textiles,
  3. calls to a telephone exchange,
  4. power outages, and
  5. atomic particles emitted from a specimen
Figure 3.14 Poisson distributions for selected values of the parameters.
3-9 Poisson Distribution

Consistent Units

It is important to use consistent units in the calculation of probabilities, means, and variances involving Poisson random variables. The following example illustrates unit conversions. For example, if the average number of flaws per millimeter of wire is 3.4, then the average number of flaws in 10 millimeters of wire is 34, and the average number of flaws in 100 millimeters of wire is 340.
3-9 Poisson Distribution

For the case of the thin copper wire, suppose that the number of flaws follows a Poisson distribution with a mean of 2.3 flaws per millimeter. Determine the probability of exactly 2 flaws in 1 millimeter of wire.

Let $X$ denote the number of flaws in 1 millimeter of wire. Then, $E(X) = 2.3$ flaws and

$$P(X = 2) = \frac{e^{-2.3}2.3^2}{2!} = 0.265$$

Determine the probability of 10 flaws in 5 millimeters of wire. Let $X$ denote the number of flaws in 5 millimeters of wire. Then, $X$ has a Poisson distribution with

$$E(X) = 5 \text{ mm} \times 2.3 \text{ flaws/mm} = 11.5 \text{ flaws}$$

Therefore,

$$P(X = 10) = e^{-11.5} \frac{11.5^{10}}{10!} = 0.113$$
Contamination is a problem in the manufacture of optical storage disks. The number of particles of contamination that occur on an optical disk has a Poisson distribution, and the average number of particles per centimeter squared of media surface is 0.1. The area of a disk under study is 100 squared centimeters. Find the probability that 12 particles occur in the area of a disk under study.

Let $X$ denote the number of particles in the area of a disk under study. Because the mean number of particles is 0.1 particles per cm$^2$

$$E(X) = 100 \text{ cm}^2 \times 0.1 \text{ particles/cm}^2 = 10 \text{ particles}$$

Therefore,

$$P(X = 12) = \frac{e^{-10}10^{12}}{12!} = 0.095$$
**Example 3-33**

The probability that zero particles occur in the area of the disk under study is

\[ P(X = 0) = e^{-10} = 4.54 \times 10^{-5} \]

Determine the probability that 12 or fewer particles occur in the area of the disk under study. The probability is

\[ P(X \leq 12) = P(X = 0) + P(X = 1) + \cdots + P(X = 12) = \sum_{i=0}^{12} \frac{e^{-10}10^i}{i!} \]
3-9 Poisson Distribution

Mean and Variance

If $X$ is a Poisson random variable with parameter $\lambda$, then

$$\mu = E(X) = \lambda \quad \text{and} \quad \sigma^2 = V(X) = \lambda \quad (3-17)$$