Instructions:

- Answer Questions 1-3 and make a table for your answers of question 1.
- Answer two and only two of Questions 4-6.
- You may use any theorem or any result that has been proven in this course. You don’t need to reprove such a theorem or a result unless you’re explicitly asked to do so.
- Each question is worth 20 marks. Be neat and organized.

1. Mark the correct statement with (√) and the false statement with (×) for each of the following:

   (1). $\text{Re}(iz) = \text{Im}z \quad \forall z \in \mathbb{C}$.
   (2). 0 is an accumulation point of the set $S = \{z \in \mathbb{C} : |z| > 1\}$.
   (3). The function $f(z) := \begin{cases} \frac{z^3 - 1}{z^2 - 1}, & \text{if } z \neq 1, \\ \frac{3}{2}, & \text{if } z = 1 \end{cases}$ is continuous at $z = 1$.
   (4). If $f(z) = u(z) + iv(z)$ satisfies the Cauchy-Riemann equations at $z_0$, then $f'(z_0)$ exists.
   (5). $\cosh(\frac{\pi i}{4}) = \frac{1}{\sqrt{2}}$.
   (6). The function $e^z : \mathbb{C} \to \mathcal{R}$, where $\mathcal{R}$ is the Riemann surface, is one-to-one.
   (7). $\int_{|z|=1} e^{-z} \, dz = 0$.
   (8). $\log(z_1 z_2) = \log z_1 + \log z_2 \quad \forall z_1, z_2 \in \mathcal{R}$, where $\log z$ is the principal value of the logarithm.
   (9). Every polynomial of degree greater than or equals zero has a root in $\mathbb{C}$.
   (10). The radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{1+(-1)^n}{2^n} z^n$ is 1.

2. Prove the following statements:

   (a). $|z| = 1$ if and only if $\frac{1}{z} = \overline{z}$.
   (b). $i^k = e^{\frac{2\pi k}{2}}$, $k \in \mathbb{Z}$.
   (c). $\int_{|z|=2} |z - 1| \, |dz| \geq 8\pi$.
   (d). The open disk $D(z_0, R) = \{z : |z - z_0| < R\}$ is an open set in $\mathbb{C}$.
3. (a). Show that the **Laurent series** of the function \( f(z) = \frac{z^2 + 2}{z^2 - 5z + 4} \) on the annulus \( 1 < |z| < 4 \) is given by

\[
\sum_{n=-\infty}^{\infty} a_n z^n
\]

where

\[
a_n = \begin{cases} 
\frac{-1}{2^{n+1}}, & n = 0, 1, 2, \ldots \\
-1, & n = -1, -2, -3, \ldots
\end{cases}
\]

(b). Use \( \epsilon - \delta \) definition of the limit to show that

\[
\lim_{z \to 2i} z^2 + 4 = 0.
\]

**Answer two** of Questions 4-6:

4. (a). Find a parametrization of the line segment \( \gamma \) from \( -\frac{\pi}{2} + i \) to \( \pi + i \).

(b). Show that \( \forall x, y \in \mathbb{R} \),

\[
\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y.
\]

(c). Use the definition of Line Integral and part (b) to show that

\[
\int_\gamma \cos z \, dz = \cosh(1) - i \sinh(1),
\]

where \( \gamma \) is the line segment given in part (a).

5. (a). State and prove the **Residue Theorem**.

(b). Use the substitution \( z = e^{i\theta}, 0 \leq \theta \leq 2\pi \), and the Residue Theorem to evaluate the integral

\[
\int_0^{2\pi} \frac{d\theta}{2 + \cos \theta}
\]

6. (a). State and prove **Cauchy’s Integral Formula**.

(b). Use Cauchy’s Integral Formula or the Residue Theorem to evaluate

\[
\int_{|z+1|=2} \frac{z^2}{4 - z^2} \, dz
\]