Instructions:

- Answer Question 1 and make a table for your answers.
- Answer any four of Questions 2-6.
- You may use any theorem or any result that has been proven in this course. You don’t need to reprove such a theorem or a result unless you’re explicitly asked to do so.
- Each question is worth 20 marks. Be neat and organized.

1. Mark the correct statement with (√) and the false statement with (×) for each of the following:

1. \( e^{iz} = e^{-iz} \quad \forall z \in \mathbb{C} \)
2. \( \arg zw = \arg z \arg w \quad \forall z, w \in \mathbb{C} \)
3. The set \( 1 < |z - 1| < 2 \) is connected but not simply connected.
4. The function \( f(z) = e^z \) is one-to-one.
5. If \( f(z) \) is continuous on a region \( G \), then \( \text{Re} f(z) \) and \( \text{Im} f(z) \) are continuous on \( G \).
6. If \( f(z) = u(z) + iv(z) \) satisfies the Cauchy-Riemann equations at \( z_0 \), then \( f'(z_0) \) exists.
7. \( 1^a = 1 \quad \forall a \in \mathbb{C} \)
8. If \( f(z) \) is analytic and real-valued on a region \( G \), then \( f \) is constant.
9. The radius of convergence of the power series \( \sum_{n=0}^{\infty} \frac{1}{2^n} z^{2n} \) is 2.
10. The function \( f(z) := e^{\frac{1}{z+1}} \) has an essential singularity at \( z = -1 \).

2. (a). Prove that the geometric series \( \sum_{n=0}^{\infty} w^n \) converges to \( \frac{1}{1-w} \) for \( |w| < 1 \).
(b). Use part (a) to show that the Laurent series of the function \( f(z) = \frac{1}{z^{3}+z} \) in the region \( 0 < |z - 1| < 1 \) is
\[ \sum_{n=0}^{\infty} [1 - \frac{1}{2n+1}](1 - z)^n. \]

3. (a). State the following theorems:
   i) Cauchy’s Theorem for Derivatives.
   ii) The Residue Theorem.
(b). Use Cauchy’s Theorem for Derivatives or the Residue Theorem to evaluate
\[ \int_{\gamma} \frac{\sin z}{z^{3}+z} \, dz, \]
where \( \gamma : z(t) = 2e^{it} - 1, \ 0 \leq t \leq 2\pi. \)
4. (a). Use the substitution \( z = e^{i\theta}, \ 0 \leq \theta \leq 2\pi \), to show that
\[
\int_0^{2\pi} \frac{d\theta}{10 - 6 \cos \theta} = \frac{1}{i} \int_{|z|=1} \frac{dz}{3z^2 - 10z + 3}.
\]

(b). Use part (a) and the Residue Theorem to evaluate
\[
\int_0^{2\pi} \frac{d\theta}{10 - 6 \cos \theta}.
\]

5. (a). Use DeMoivre’s Theorem to find the solutions of the equation \( z^3 = -1 \) (Write each solution in the form \( x + iy \)).

(b). Prove the Fundamental Theorem of Algebra: “Any nonconstant polynomial
\[
p(z) = a_n z^n + \cdots + a_1 z + a_0, \ a_n \neq 0,
\]
has at least one root in \( \mathbb{C} \).

(c). Show that if the polynomial \( p(z) \) in part (b) has degree \( n \geq 1 \), then \( p(z) \) has exactly \( n \) roots (including repeated roots).

6. (a). Prove the following inequality:
\[
|z_1 - z_2| \geq |z_1| - |z_2| \quad \forall \ z_1, z_2 \in \mathbb{C}.
\]

(b). Use Cauchy’s Theorem for Derivatives and part(a) to prove that if \( f(z) \) is analytic in the closed disk \( \overline{D}(0, R) = \{z \in \mathbb{C}: |z| \leq R\} \) and if \( |f(z)| \leq M \ \forall z \in \overline{D}(0, R), \) then
\[
|f^n(\xi)| \leq \frac{MR(n!)}{(R - |\xi|)^{n+1}} \quad \forall \ \xi : |\xi| < R.
\]

(c). Evaluate
\[
\int_{|z|=1} |z + 1| \ |dz|.
\]