Chapter 6

Counting

Eng. Huda M. Dawoud

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Section 1: The Basics of Counting

**The Product Rule** Suppose that a procedure can be broken down into a sequence of two tasks. If there are $n_1$ ways to do the first task and for each of these ways of doing the first task, there are $n_2$ ways to do the second task, then there are $n_1n_2$ ways to do the procedure.

**The Sum Rule** If a task can be done either in one of $n_1$ ways or in one of $n_2$ ways, where none of the set of $n_1$ ways is the same as any of the set of $n_2$ ways, then there are $n_1 + n_2$ ways to do the task.

**The Subtraction Rule** If a task can be done in either $n_1$ ways or $n_2$ ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

**The Division Rule** There are $n/d$ ways to do a task if it can be done using a procedure that can be carried out in $n$ ways, and for every way $w$, exactly $d$ of the $n$ ways correspond to way $w$.

1. There are 18 mathematics majors and 325 computer science majors at a college.

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

b) In how many ways can one representative be picked who is either mathematics major or a computer science major?

Answer:
This problem illustrates the difference between the product rule and the sum rule. If we must make one choice and then another choice, the product rule applies, as in part (a). If we must make one choice or another choice, the sum rule applies, as in part (b).

a) $P($mathematics major$) \times P($computer science major$) = 18 \times 325$

b) $P($mathematics major$) + P($computer science major$) = 18 + 325$
2. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

Answer:

\[ 27 \times 37 \]

3. A multiple-choice test contains 10 questions. There are four possible answers for each question.

a) In how many ways can a student answer the questions on the test if the student answers every question?

b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

Answer:

a) \[ 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^{10} \]

b) \[ 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^{10} \] (we have another possible answer which is to leave answer blank)

4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

Answer:

\[ 12 \text{ color} \times 2 \text{ versions} \times 3 \text{ sizes} = 72 \text{ different shirt} \]
6. There are four major auto routes from Boston to Detroit and six from Detroit to Los Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit?

Answer:

![Diagram showing routes between Boston, Detroit, and Los Angeles with 4 routes from Boston to Detroit, 6 from Detroit to Los Angeles, and 24 total routes via Detroit.]

Routes from Boston to Los Angeles = 4 x 6 = 24

7. How many different three-letter initials can people have?

Answer:

26 x 26 x 26

8. How many different three-letter initials with none of the letters repeated can people have?

Answer:

26 x 25 x 24
13. How many bit strings with length not exceeding \( n \), where \( n \) is a positive integer, consist entirely of 1's, not counting the empty string?

Answer:

This is a trick question, since it is easier than one might expect. Since the string is given to consist entirely of 1's, there is nothing to choose except the length. Since there are \( n + 1 \) possible lengths not exceeding \( n \) (if we include the empty string, of length 0), the answer is simply \( n + 1 \). Note that the empty string consists—vacuously—entirely of 1's.

14. How many bit strings of length \( n \), where \( n \) is a positive integer, start and end with 1s?

Answer:

\[ 2 \times (n-2) \]

21. How many positive integers between 50 and 100
a) are divisible by 7? Which integers are these?

b) are divisible by 11? Which integers are these?

c) are divisible by both 7 and 11? Which integers are these?

Answer:

a) First I know that the smallest multiple of 7 that is larger than or equal to 50 is \( 56 = 7 \times 8 \).
On the other hand, the largest multiple of 7 that is less than or equal to 100 is \( 98 = 7 \times 14 \).
Now the numbers between 50 and 100 that are divisible by 7 are:
\[ 98, 91, 84, 77, 70, 63, 56 \]
\[ 7 \times 14, 7 \times 13, 7 \times 12, 7 \times 11, 7 \times 10, 7 \times 9, 7 \times 8 \]
We can count them as \( 14 - 7 = 7 \Rightarrow [100/7] - [50/7] \)
b) \( \lfloor \frac{100}{11} \rfloor - \lfloor \frac{50}{11} \rfloor = 9 - 4 = 5 \)

99, 88, 77, 66, 55

c) A number is divisible by both 7 and 11 if and only if it is divisible by their least common multiple, which is 77. Obviously there is only one such number between 50 and 100, namely 77. We could also work this out as we did in the previous parts: \( \lfloor \frac{100}{77} \rfloor - \lfloor \frac{50}{77} \rfloor = 1 - 0 = 1 \). Note also that the intersection of the sets we found in the previous two parts is precisely what we are looking for here.

23. How many positive integers between 100 and 999 inclusive

a) are divisible by 7?

b) are odd?

c) have the same three decimal digits?

d) are not divisible by 4?

e) are divisible by 3 or 4?

f) are not divisible by either 3 or 4?

g) are divisible by 3 but not by 4?

h) are divisible by 3 and 4?

Answer:

\[
\begin{align*}
\text{a) } & \lfloor \frac{999}{7} \rfloor - \lfloor \frac{99}{7} \rfloor = 142 - 14 = 128 \\
\text{b) } & \text{We know that any number divisible by 2 is an even number, so we calculate} \\
& \lfloor \frac{999}{2} \rfloor = 499 \text{ even number not exceeding 999 (Note that 100 is included)} \\
& \text{So we have } 999 - 499 = 500 \text{ odd number not exceeding 999} \\
& \lfloor \frac{99}{2} \rfloor = 49 \text{ even number not exceeding 99}
\end{align*}
\]
And we have $99 - 49 = 50$ odd number not exceeding 99
$500 - 50 = 450$ odd number between 100 and 999

c) There are just 9 possible digits that a three-digit number can start
with. If all of its digits are to be the same, then there is no choice after
the leading digit has been specified. Therefore there are 9 such
numbers.

d) $\left\lfloor \frac{999}{4} \right\rfloor - \left\lfloor \frac{99}{4} \right\rfloor = 249 - 24 = 225$ number that is divisible by 4
$999 - 99 = 900$
$900 - 225 = 675$ number that is not divisible by 4

e) There are $\left\lfloor \frac{999}{3} \right\rfloor - \left\lfloor \frac{99}{3} \right\rfloor = 300$ three-digit numbers divisible by
3, and $\left\lfloor \frac{999}{4} \right\rfloor - \left\lfloor \frac{99}{4} \right\rfloor = 225$ three-digit numbers divisible by 4.
Moreover there are $\left\lfloor \frac{999}{12} \right\rfloor - \left\lfloor \frac{99}{12} \right\rfloor = 75$ numbers divisible by
both 3 and 4, i.e., divisible by 12. In order to count each number
divisible by 3 or 4 once and only once, we need to add the number of
numbers divisible by 3 to the number of numbers divisible by 4, and
then subtract the number of numbers divisible by both 3 and 4 so as not
to count them twice. Therefore the answer is $300 + 225 - 75 = 450$.

f) $900 - 450 = 450$ three-digit integers that are not divisible by either 3
or 4.

g) We saw in part (e) that there are 300 three-digit numbers divisible
by 3 and that 75 of them are also divisible by 4. The remainder of those
300 numbers, therefore, are not divisible by 4. Thus the answer is $300 - 75 = 225$.

h) $\left\lfloor \frac{999}{12} \right\rfloor - \left\lfloor \frac{99}{12} \right\rfloor = 75$ numbers divisible by both 3 and 4.

25. How many strings of three decimal digits

a) do not contain the same digit three times?

b) begin with an odd digit?
c) have exactly two digits that are 4s?

Answer:

a) Clearly there are 10 strings that consist of the same digit three times (000, 111, ..., 999). Therefore there are 1000 - 10 = 990 strings that do not.

b) 5 * 10 * 10 = 500

c) Here we need to choose the position of the digit that is not a 4 (3 ways) and choose that digit (9 ways). Therefore there are 3 * 9 = 27 such strings.
Section 2: The Pigeonhole Principle

**THE PIGEONHOLE PRINCIPLE** If \( k \) is a positive integer and \( k + 1 \) or more objects are placed into \( k \) boxes, then there is at least one box containing two or more of the objects.

**THE GENERALIZED PIGEONHOLE PRINCIPLE** If \( N \) objects are placed into \( k \) boxes, then there is at least one box containing at least \( \lceil N/k \rceil \) objects.

4. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

   a) How many balls must she select to be sure of having at least three balls of the same color?

   b) How many balls must she select to be sure of having at least three blue balls?

   Answer:

   a) \( \lceil x/2 \rceil = 3 \), \( x = 5 \)

   b) 13 ball

15. How many numbers must be selected from the set \( \{1, 2, 3, 4, 5, 6\} \) to guarantee that at least one pair of these numbers add up to 7?

   Answer:

   We can apply the pigeonhole principle by grouping the numbers cleverly into pairs (subsets) that add up to 7, namely \( \{1,6\} \), \( \{2,5\} \), and \( \{3,4\} \). If we select four numbers from the set \( \{1,2,3,4,5,6\} \), then at least two of them must fall within the same subset, since there are only three subsets. Two numbers in the same subset are the desired pair that add up to 7. We also need to point out that choosing three numbers is not enough, since we could choose \( \{1, 2, 3\} \), and no pair of them add up to more than 5.
Section 3: Permutations and Combinations

If \( n \) is a positive integer and \( r \) is an integer with \( 1 \leq r \leq n \), then there are

\[
P(n, r) = n(n-1)(n-2) \cdots (n-r+1)
\]

\( r \)-permutations of a set with \( n \) distinct elements.

If \( n \) and \( r \) are integers with \( 0 \leq r \leq n \), then \( P(n, r) = \frac{n!}{(n-r)!} \).

The number of \( r \)-combinations of a set with \( n \) elements, where \( n \) is a nonnegative integer and \( r \) is an integer with \( 0 \leq r \leq n \), equals

\[
C(n, r) = \frac{n!}{r!(n-r)!}.
\]

Let \( n \) and \( r \) be nonnegative integers with \( r \leq n \). Then \( C(n, r) = C(n, n-r) \).

3. How many permutations of \( \{a, b, c, d, e, f, g\} \) end with \( a \)?

Answer:

If we want the permutation to end with \( a \), then we may as well forget about the \( a \), and just count the number of permutations of \( \{b, c, d, e, f, g\} \). Each permutation of these 6 letters, followed by \( a \), will be a permutation of the desired type, and conversely. Therefore the answer is \( P(6, 6) = 6! = 720 \).
5. Find the value of each of these quantities.
   a) \( P(6, 3) \)
   b) \( P(6, 5) \)

   Answer:
   a) \[ P(6,3)=6\cdot5\cdot4=120 \]
   b) \[ P(6,5)=6!=720 \]

6. Find the value of each of these quantities.
   a) \( C(5, 1) \)
   b) \( C(5, 3) \)

   Answer:
   a) \[ 5! / (1! \cdot 4!) = 5\cdot4! / 4! = 5 \]
   b) \[ 5! / (3! \cdot 2!) = 5\cdot4\cdot3! / 3! \cdot 2! = 20 / 2 = 10 \]

11. How many bit strings of length 10 contain
   a) exactly four 1s?
   b) at most four 1s?
   c) at least four 1s?
   d) an equal number of 0s and 1s?

   Answer:
   a) To specify a bit string of length 10 that contains exactly four l's, we simply need to choose the four positions that contain the l's. There are \( C(10,4) = 210 \) ways to do that.
   b) To contain at most four l's means to contain four l's, three l's, two l's, one 1, or no l's. Reasoning as in part (a), we see that there are \( C(10, 4)+C(10, 3)+C(10, 2)+C(10, 1)+C(10, 0) = 210+120+45+10+1 = 386 \) such strings.
c) To contain at least four l's means to contain four l's, five l's, six l's, seven l's, eight l's, nine l's, or ten l's. Reasoning as in part (b), we see that there are \( C(10, 4) + C(10, 5) + C(10, 6) + C(10, 7) + C(10, 8) + C(10, 9) + C(10, 10) = 210 + 252 + 210 + 120 + 45 + 10 + 1 = 848 \) such strings.

d) To have an equal number of O's and l's in this case means to have five l's. Therefore the answer is \( C(10, 5) = 252 \).

21. How many permutations of the letters ABCDEFG contain

a) the string BCD?

b) the string CFGA?

c) the strings BA and GF?

d) the strings ABC and DE?

e) the strings ABC and CDE?

f) the strings CBA and BED?

Answer:

a) If BCD is to be a substring, then we can think of that block of letters as one superletter, and the problem is to count permutations of five items—the letters A, E, F, and G, and the superletter BCD. Therefore the answer is \( P(5, 5) = 5! = 120 \).

b) Reasoning as in part (a), we see that the answer is \( P(4,4) = 4! = 24 \).

c) As in part (a), we glue BA into one item and glue GF into one item. Therefore we need to permute five items, and there are \( P(5, 5) = 5! = 120 \) ways to do it.

d) This is similar to part (c). Glue ABC into one item and glue DE into one item, producing four items, so the answer is \( P(4,4) = 4! = 24 \).
e) If both ABC and CDE are substrings, then ABCDE has to be a substring. So we are really just permuting three items: ABCDE, F, and G. Therefore the answer is $P(3,3) = 3! = 6$.

f) There are no permutations with both of these substrings, since B cannot be followed by both A and E at the same time.

31. The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the English alphabet contain

a) exactly one vowel?

b) exactly two vowels?

c) at least one vowel?

d) at least two vowels?

Answer:

We need to be careful here, because strings can have repeated letters.

a) We need to choose the position for the vowel, and this can be done in 6 ways. Next we need to choose the vowel to use, and this can be done in 5 ways. Each of the other five positions in the string can contain any of the 21 consonants, so there are $21^5$ ways to fill the rest of the string. Therefore the answer is $6 \cdot 5 \cdot 21^5 = 122,523,030$.

b) We need to choose the position for the vowels, and this can be done in $C(6, 2) = 15$ ways (we need to choose two positions out of six). We need to choose the two vowels (5 ways). Each of the other four positions in the string can contain any of the 21 consonants, so there are 214 ways to fill the rest of the string. Therefore the answer is $15 \cdot 5^2 \cdot 21^4 = 72,930,375$. 
c) The best way to do this is to count the number of strings with no vowels and subtract this from the total number of strings. We obtain $26^6 - 21^6 = 223,149,655$.

d) As in part (c), we will do this by subtracting from the total number of strings, the number of strings with no vowels and the number of strings with one vowel (this latter quantity having been computed in part (a)). We obtain $26^6 - 21^6 - 6 \cdot 5 \cdot 21^5 = 223149655 - 122523030 = 100,626,625$. 
Section 4: Binomial Coefficients and Identities

**THE BINOMIAL THEOREM**  
Let $x$ and $y$ be variables, and let $n$ be a nonnegative integer. Then

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$ 

**PASCAL’S IDENTITY**  
Let $n$ and $k$ be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$ 

1. Find the expansion of $(x + y)^4$

   a) using combinatorial reasoning, as in Example 1.
   
   b) using the binomial theorem.

   **Answer:**

   a) $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
   
   b) $\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$

7. What is the coefficient of $x^9$ in $(2 - x)^{19}$?

   **Answer:**

   $$(2 + (-x))^{19} \text{ is } \binom{19}{9} 2^{10} (-x)^9. \text{ Therefore}$$

   the coefficient is $\binom{19}{9} 2^{10} (-x)^9 = -2^{10} \binom{19}{9} = -94,595,072.$
13. What is the row of Pascal’s triangle containing the binomial coefficients \( \binom{9}{k} \cdot 2^{10}, 0 \leq k \leq 9 \)?

Answer:

\[
1 \ 9 \ 36 \ 84 \ 126 \ 126 \ 84 \ 36 \ 9 \ 1.
\]