1.7 Determine the equivalent spring constant of the system shown in Fig. 1.67.

\[ \frac{1}{k_{eq}} = \frac{1}{2k_4} + \frac{1}{k_2} + \frac{1}{2k_3} \quad ; \quad k_{eq} = \left( \frac{2k_1 k_2 k_3}{k_1 k_2 + 2k_1 k_3 + k_1 k_2} \right) \]

\[ \frac{1}{k_{eq}} = \frac{1}{k_{eq} + k_4} + \frac{1}{k_5} \]

\[ k_{eq} = \frac{k_5 (k_{eq} + k_4)}{k_5 + k_4 + k_{eq}} = \frac{k_2 k_3 k_4 k_5 + 2k_1 k_3 k_4 k_5 + k_1 k_2 k_4 k_5 + 2k_1 k_1 k_3 k_5}{k_4 k_3 k_4 + k_2 k_3 k_5 + 2k_1 k_1 k_3 k_4 + 2k_1 k_3 k_5 + k_1 k_2 k_4 + k_1 k_2 k_3 + 2k_1 k_5 k_3} \]
1.9 In Fig. 1.69, find the equivalent spring constant of the system in the direction of $\theta$.

Equivalence of potential energies gives

$$\frac{1}{2} k_{t1} \theta^2 + \frac{1}{2} k_{t2} \theta^2 + \frac{1}{2} k_1 (\theta l_1)^2 + \frac{1}{2} k_2 (\theta l_1)^2 + \frac{1}{2} k_3 (\theta l_2)^2 = \frac{1}{2} k_{eq} \theta^2$$

$$\therefore k_{eq} = k_{t1} + k_{t2} + k_1 l_1^2 + k_2 l_1^2 + k_3 l_2^2$$

**FIGURE 1.69**
1.11 A machine of mass $m = 500$ kg is mounted on a simply supported steel beam of length $l = 2$ m having a rectangular cross section (depth = 0.1 m, width = 1.2 m) and Young’s modulus $E = 2.06 \times 10^{11}$ N/m$^2$. To reduce the vertical deflection of the beam, a spring of stiffness $k$ is attached at mid-span, as shown in Fig. 1.71. Determine the value of $k$ needed to reduce the deflection of the beam by

a. 25 percent of its original value.

b. 50 percent of its original value.

c. 75 percent of its original value.

Assume that the mass of the beam is negligible.
For simply supported beam, for load at middle,

\[ k_1 = \frac{48 \cdot E \cdot I}{l^3} = \frac{48 \cdot (2.06 \times 10^{11}) \cdot (10^{-4})}{8} \]

\[ = 12.36 \times 10^7 \text{ N/m} \quad \text{where} \quad I = \frac{1}{12} \cdot (1.2) \cdot (0.1)^3 = 10^{-4} \text{ m}^4. \]

\[ \delta_1 = \text{original deflection} = \frac{m \cdot g}{k_1} = \frac{500 \times 9.81}{12.36 \times 10^7} = 396.8447 \times 10^{-7} \text{ m} \]

When spring \( k \) is added, \( k_{eq} = k + k_1 \)

(a) New deflection \( \frac{m \cdot g}{k_{eq}} = \frac{\delta_1}{4} \); \( k_{eq} = \frac{4 \cdot m \cdot g}{\delta_1} = 4 \cdot k_1 \)

\[ \therefore k = 3 \cdot k_1 = 37.08 \times 10^7 \text{ N/m} \]

(b) New deflection \( \frac{m \cdot g}{k_{eq}} = \frac{\delta_1}{2} \); \( k_{eq} = \frac{2 \cdot m \cdot g}{\delta_1} = 2 \cdot k_1 \)

\[ \therefore k = k_1 = 12.36 \times 10^7 \text{ N/m} \]

(c) New deflection \( \frac{m \cdot g}{k_{eq}} = \frac{3}{4} \delta_1 \); \( k_{eq} = \frac{4 \cdot m \cdot g}{3 \cdot \delta_1} = \frac{4}{3} \cdot k_1 \)

\[ \therefore k = \frac{1}{3} \cdot k_1 = 4.12 \times 10^7 \text{ N/m} \]

\[ = k + k_1 \]
1.43 A composite propeller shaft, made of steel and aluminum, is shown in Fig. 1.92.

a. Determine the torsional spring constant of the shaft.

b. Determine the torsional spring constant of the composite shaft when the inner diameter of the aluminum tube is 5 cm instead of 10 cm.
4) The steel and aluminum hollow shafts can be treated as two torsional springs in parallel.

For a hollow shaft,

\[ k_t = \frac{\pi G}{32\ell} (D^4 - d^4) \]

For the steel shaft, \( G = 80 \times 10^9 \) Pa, \( \ell = 5 \) m, \( D = 0.25 \) m, \( d = 0.15 \) m, and hence

\[ k_{t_1} = \frac{\pi (8 \times 10^{10})}{32 (5)} (0.25^4 - 0.15^4) = 5.34072 \times 10^8 \text{ N·m/rad} \]

\nu) For the aluminum shaft, \( G = 26 \times 10^9 \) Pa, \( \ell = 5 \) m, \( D = 0.15 \) m, \( d = 0.1 \) m, and hence

\[ k_{t_2} = \frac{\pi (26 \times 10^9)}{32 (5)} (0.15^4 - 0.10^4) = 0.207395 \times 10^8 \text{ N·m/rad} \]

\[ k_{eq} = k_{t_1} + k_{t_2} = 5.34072 \times 10^8 + 0.207395 \times 10^8 = 5.54811 \times 10^8 \text{ N·m/rad} \]

b) With \( G = 26 \times 10^9 \) Pa, \( \ell = 5 \) m, \( D = 0.15 \) m and \( d = 0.05 \) m,

\[ k_{t_2} = \frac{\pi (26 \times 10^9)}{32 (5)} (0.15^4 - 0.05^4) = 0.255255 \times 10^6 \text{ N·m/rad} \]

\[ k_{eq} = k_{t_1} + k_{t_2} = 5.34072 \times 10^6 + 0.255255 \times 10^6 = 5.595975 \times 10^6 \text{ N·m/rad} \]
1.49 In Fig. 1.96 find the equivalent mass of the rocker arm assembly with respect to the $x$ coordinate.

\[ \theta = \frac{x}{b}, \quad x_1 = \frac{x}{\frac{a}{b}}. \]

From equivalence of kinetic energies,

\[ \frac{1}{2} m_{eq} \ddot{x}^2 = \frac{1}{2} m_1 \ddot{x}_1^2 + \frac{1}{2} m_2 \ddot{x}_2^2 + \frac{1}{2} J_0 \dot{\theta}^2 \]

\[ m_{eq} = m_1 \left( \frac{a}{b} \right)^2 + m_2 + J_0 \left( \frac{1}{b} \right)^2. \]
A massless bar of length 1 m is pivoted at one end and subjected to a force $F$ at the other end. Two translational dampers, with damping constants $c_1 = 10 \text{ N-s/m}$ and $c_2 = 15 \text{ N-s/m}$ are connected to the bar as shown in Fig. 1.109. Determine the equivalent damping constant, $c_{\text{eq}}$, of the system so that the force $F$ at point $A$ can be expressed as $F = c_{\text{eq}} v$, where $v$ is the linear velocity of point $A$. 
1.59 Develop an expression for the damping constant of the rotational damper shown in Fig. 1.105 in terms of $D$, $d$, $l$, $h$, $\omega$, and $\mu$, where $\omega$ denotes the constant angular velocity of the inner cylinder, and $d$ and $h$ represent the radial and axial clearances between the inner and outer cylinders.
Tangential velocity of inner cylinder = \( \frac{D}{2} \omega \)

For small \( d \), rate of change of velocity of fluid is

\[
\frac{dv}{dr} = \frac{D \omega}{2d}
\]

Shear stress between cylinders is

\[
\tau = \mu \frac{dv}{dr} = \mu \frac{D \omega}{2d}
\]

and shear force is

\[
F = \tau \cdot \text{Area} = \tau \pi D(l-h) = \frac{\pi \mu D^2 \omega (l-h)}{2d}
\]

Torque developed = \( M_{t1} = F \cdot \frac{D}{2} \)

For small \( h \), rate of change of velocity of fluid in vertical direction is

\[
\frac{dv}{dy} = \frac{r \omega}{h}
\]

Shear stress is

\[
\tau = \mu \frac{dv}{dy} = \frac{\mu r \omega}{h}
\]

Force on area \( dA = dF = \tau \ dA \)

Torque between bottom surfaces of cylinders is

\[
M_{t2} = \iiint dM_{t2} \cdot dA \quad \text{where} \quad dM_{t2} = dF \cdot r = \frac{\mu r^3 \omega}{h} \ dr \ d\theta \quad \text{area}
\]

\[
i.e., \quad M_{t2} = \frac{\mu \omega}{h} \int_{0}^{\frac{D}{2}} r^3 \ dr \int_{0}^{2\pi} \ d\theta = \frac{\mu \omega \pi D^4}{64h}
\]

Total torque = \( M_t = M_{t1} + M_{t2} = \frac{\pi \mu D^3 \omega (l-h)}{4d} + \frac{\pi \mu \omega D^4}{64h} \)

Expressing \( M_t \) as \( C_t \cdot \omega = C_t \omega D/2 \), we get damping constant:

\[
C_t = \frac{\pi \mu D^2 (l-h)}{2d} + \frac{\pi \mu D^3}{32h}
\]
1.82 A machine is subjected to the motion \( x(t) = A \cos(50t + \alpha) \) mm. The initial conditions are given by \( x(0) = 3 \text{ mm} \) and \( \dot{x}(0) = 1.0 \text{ m/s} \).

a. Find the constants \( A \) and \( \alpha \).

b. Express the motion in the form \( x(t) = A_1 \cos \omega t + A_2 \sin \omega t \), and identify the constants \( A_1 \) and \( A_2 \).

\[
(x) \quad x(t) = \frac{A}{1000} \cos (50t + \alpha) \text{ m} \quad \text{where} \quad A \text{ is in mm} \quad ---- \quad (E_1)
\]

\[
x(0) = \frac{A}{1000} \cos \alpha = 0.003 \quad , \quad A \cos \alpha = 3 \quad ---- \quad (E_2)
\]

\[
\dot{x}(0) = -\frac{50A}{1000} \sin \alpha = 1 \quad , \quad A \sin \alpha = -20 \quad ---- \quad (E_3)
\]

\[
A = \left\{ (A \cos \alpha)^2 + (A \sin \alpha)^2 \right\}^{1/2} = 20.2237 \text{ mm}
\]

\[
\alpha = \tan^{-1} \left( \frac{A \sin \alpha}{A \cos \alpha} \right) = \tan^{-1} (-6.6667) = -81.4692^\circ = -1.4219 \text{ rad}
\]

\[
x(t) = 20.2237 \cos (50t - 1.4219) \quad \text{mm}
\]

(b) \( \cos(A+B) = \cos A \cos B - \sin A \sin B \)

\(\) Eq. \((E_1)\) can be expressed as \( x(t) = A \cos 50t \cos \alpha - A \sin 50t \sin \alpha \)

\[
= A_1 \cos \omega t + A_2 \sin \omega t
\]

where \( \omega = 50 \), \( A_1 = A \cos \alpha \), \( A_2 = -A \sin \alpha \)

\[
\therefore \ x(t) = (3 \cos 50t + 20 \sin 50t) \text{ mm}
\]
1.83 Show that any linear combination of \( \sin \omega t \) and \( \cos \omega t \) such that \( x(t) = A_1 \cos \omega t + A_2 \sin \omega t \) \((A_1, A_2 = \text{constants})\) represents a simple harmonic motion.

\[
x(t) = A_1 \cos \omega t + A_2 \sin \omega t
\]
\[
\frac{dx}{dt}(t) = -A_1 \omega \sin \omega t + A_2 \omega \cos \omega t, \quad \frac{d^2x}{dt^2} = -A_1 \omega^2 \cos \omega t - A_2 \omega^2 \sin \omega t
\]
\[
\frac{d^2x}{dt^2} = -\omega^2 x(t) \quad \text{where} \quad \omega^2 \quad \text{is a constant}
\]
Hence \( x(t) \) is a simple harmonic motion.
1.84 Find the sum of the two harmonic motions $x_1(t) = 5 \cos(3t + 1)$ and $x_2(t) = 10 \cos(3t + 2)$. Use:

a. Trigonometric relations
b. Vector addition
c. Complex-number representation
(a) Using trigonometric relations:
\[ x_1(t) = 5 \left( \cos 3t \cos 1 - \sin 3t \sin 1 \right) \]
\[ x_2(t) = 10 \left( \cos 3t \cos 2 - \sin 3t \sin 2 \right) \]
\[ x(t) = x_1(t) + x_2(t) = \cos 3t \left( 5 \cos 1 + 10 \cos 2 \right) - \sin 3t \left( 5 \sin 1 + 10 \sin 2 \right) \]
If \[ x(t) = A \cos (\omega t + \alpha) = A \cos \omega t \cos \alpha - A \sin \omega t \sin \alpha \],
\[ \omega = 3, \quad A \cos \alpha = 5 \cos 1 + 10 \cos 2 = -1.4599, \]
\[ A \sin \alpha = 5 \sin 1 + 10 \sin 2 = 13.3003 \]
\[ A = \sqrt{(A \cos \alpha)^2 + (A \sin \alpha)^2} = 13.3802 \]
\[ \alpha = \tan^{-1} \left( \frac{A \sin \alpha}{A \cos \alpha} \right) = \tan^{-1} (-9.1104) = 96.2640^\circ = 1.68 \text{ rad} \]
Angle between \( x_1(t) \) and \( x(t) \) is \( 96.2640^\circ - 57.3^\circ = 38.964^\circ \)

(b) Using vector addition:

For an arbitrary value of \( (\omega t + 1) \), harmonic motions \( x_1(t) \) and \( x_2(t) \) can be shown as in the figure. From vector addition, we find
\[ x(t) = 13.38 \cos (\omega t + 1.68) \]
(C) Using complex numbers:

\[ x_1(t) = \text{Re} \left\{ A_1 e^{i(\omega t + 1)} \right\} = \text{Re} \left\{ 5 e^{i(\omega t + 1)} \right\} \]
\[ x_2(t) = \text{Re} \left\{ A_2 e^{i(\omega t + 2)} \right\} = \text{Re} \left\{ 10 e^{i(\omega t + 2)} \right\} \]

If \( x(t) = \text{Re} \left\{ A e^{i(\omega t + \alpha)} \right\} \),

\[ A \cos(3t + \alpha) = A_1 \cos(3t + 1) + A_2 \cos(3t + 2) \]

i.e.

\[ A (\cos 3t \cos \alpha - \sin 3t \sin \alpha) = 5 (\cos 3t \cdot \cos 1 - \sin 3t \cdot \sin 1) \]
\[ + 10 (\cos 3t \cdot \cos 2 - \sin 3t \cdot \sin 2) \]

i.e. \( A \cos \alpha = 5 \cos 1 + 10 \cos 2 \), \( A \sin \alpha = 5 \sin 1 + 10 \sin 2 \)

\[ A = 13.3802 \quad \alpha = 1.68 \text{ rad} \]

\[ x(t) = \text{Re} \left\{ 13.3802 e^{i(3t + 1.68)} \right\} \]
Find the sum of the two harmonic motions \( x_1(t) = 10 \cos \omega t \) and \( x_2(t) = 15 \cos(\omega t + 2) \).

**Solution: Method 1: By using trigonometric relations:** Since the circular frequency is the same for both \( x_1(t) \) and \( x_2(t) \), we express the sum as

\[
x(t) = A \cos(\omega t + \alpha) = x_1(t) + x_2(t)
\]  
(E.1)

That is,

\[
A(\cos \omega t \cos \alpha - \sin \omega t \sin \alpha) = 10 \cos \omega t + 15 \cos(\omega t + 2)
\]

\[
= 10 \cos \omega t + 15(\cos \omega t \cos 2 - \sin \omega t \sin 2) \tag{E.2}
\]

\[
\cos \omega t(A \cos \alpha) - \sin \omega t(A \sin \alpha) = \cos \omega t(10 + 15 \cos 2) - \sin \omega t(15 \sin 2)
\]

By equating the corresponding coefficients of \( \cos \omega t \) and \( \sin \omega t \) on both sides, we obtain

\[
A \cos \alpha = 10 + 15 \cos 2
\]

\[
A \sin \alpha = 15 \sin 2
\]

\[
A = \sqrt{(10 + 15 \cos 2)^2 + (15 \sin 2)^2}
\]

\[
= 14.1477
\]

and

\[
\alpha = \tan^{-1}\left(\frac{15 \sin 2}{10 + 15 \cos 2}\right) = 74.5963^\circ
\]
Method 2: By using vectors: For an arbitrary value of $\omega t$, the harmonic motions $x_1(t)$ and $x_2(t)$ can be denoted graphically as shown in Fig. 1.51. By adding them vectorially, the resultant vector $x(t)$ can be found to be

$$x(t) = 14.1477 \cos(\omega t + 74.5963^\circ)$$  \hspace{1cm} (E.6)

Method 3: By using complex-number representation: The two harmonic motions can be denoted in terms of complex numbers:

$$x_1(t) = \text{Re}[A_1 e^{i\omega t}] = \text{Re}[10 e^{i\omega t}]$$

$$x_2(t) = \text{Re}[A_2 e^{i(\omega t + 2)}] = \text{Re}[15 e^{i(\omega t + 2)}]$$  \hspace{1cm} (E.7)

The sum of $x_1(t)$ and $x_2(t)$ can be expressed as

$$x(t) = \text{Re}[A e^{i(\omega t + \alpha)}]$$  \hspace{1cm} (E.8)

where $A$ and $\alpha$ can be determined using Eqs. (1.47) and (1.48) as $A = 14.1477$ and $\alpha = 74.5963^\circ$. 

![Diagram showing vector addition of two harmonic motions](image)
1.85 If one of the components of the harmonic motion \( x(t) = 10 \sin(\omega t + 60^\circ) \) is \( x_1(t) = 5 \sin(\omega t + 30^\circ) \), find the other component.

\[
x(t) = 10 \sin(\omega t + 60^\circ) = x_1(t) + x_2(t)
\]

where \( x_1(t) = 5 \sin(\omega t + 30^\circ) \) and \( x_2(t) = A \sin(\omega t + \alpha^\circ) \)

\[
10 \left( \sin \omega t \cos 60^\circ + \cos \omega t \sin 60^\circ \right) = 5 \left( \sin \omega t \cos 30^\circ + \cos \omega t \sin 30^\circ \right) + A \left( \sin \omega t \cos \alpha^\circ + \cos \omega t \sin \alpha^\circ \right)
\]

\[
10 \cos 60^\circ = 5 \cos 30^\circ + A \cos \alpha^\circ \quad ; \quad A \cos \alpha^\circ = 0.6699
\]

\[
10 \sin 60^\circ = 5 \sin 30^\circ + A \sin \alpha^\circ \quad ; \quad A \sin \alpha^\circ = 6.1603
\]

\[
A = \sqrt{0.6699^2 + 6.1603^2} = 6.1966
\]

\[
\alpha = \tan^{-1} \left( \frac{6.1603}{0.6699} \right) = 83.7938^\circ
\]

\[
x_2(t) = 6.1966 \sin (\omega t + 83.7938^\circ)
\]
A harmonic motion has an amplitude of 0.05 m and a frequency of 10 Hz. Find its period, maximum velocity, and maximum acceleration.

\[ A = 0.05 \text{ m}, \quad \omega = 10 \text{ Hz} = 62.832 \text{ rad/sec} \]

\[ \text{period} = T = \frac{2\pi}{\omega} = \frac{2\pi}{62.832} = 0.1 \text{ sec} \]

\[ \text{maximum velocity} = A\omega = 0.05 \times 62.832 = 3.1416 \text{ m/s} \]

\[ \text{maximum acceleration} = A\omega^2 = 0.05 \times (62.832)^2 = 197.393 \text{ m/s}^2 \]
The maximum amplitude and the maximum acceleration of the foundation of a centrifugal pump were found to be \( x_{\text{max}} = 0.25 \text{ mm} \) and \( \dot{x}_{\text{max}} = 0.4g \). Find the operating speed of the pump.

\[
x = A \cos \omega t, \quad x_{\text{max}} = A = 0.25 \text{ mm}, \quad \ddot{x} = -\omega^2 A \cos \omega t
\]

\[
\dot{x}_{\text{max}} = A \omega^2 = 0.4g = 3924 \text{ mm/s}^2; \quad \omega^2 = 2924/A = 15696 \text{ (rad/s)}^2
\]

Operating speed of pump = \( \omega = 125.2837 \text{ rad/s} = 19.9395 \text{ rpm} \)