Chapter Outline

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Springs Classification

Springs may be classified as:

1. wire springs
   a. helical springs of round or square wire, made to resist and deflect under tensile, compressive, or torsional loads.

2. flat springs
   a. cantilever and elliptical types,
   b. wound motor- or clock-type power springs
   c. flat spring washers (Belleville springs)

3. special shaped springs
Springs Configurations

Helical Compression Springs

- Standard—constant rate
- Variable pitch—variable rate
- Barrel
- Hourglass
- Conical
Springs Configurations

(b) Helical extension springs. Pull—wide load and deflection range—round or rectangular wire, constant rate.

(c) Drawbar springs. Pull—uses compression spring and drawbars to provide extension pull with fail-safe, positive stop.

(d) Torsion springs. Twist—round or rectangular wire—constant rate.
Springs Configurations

Spring Washers

- Belleville
- Wave
- Slotted
- Finger
- Curved
Springs Configurations

(f) Volute spring. *Push*—may have an inherently high friction damping.

(g) Beam springs. *Push* or *Pull*—wide load but low deflection range—rectangular or shaped cantilever, or simply supported.

(h) Power or motor springs. *Twist*—exerts torque over many turns. Shown in, and removed from, retainer.

(i) Constant Force. *Pull*—long deflection at low or zero rate.
10–1 Stresses in Helical Springs

- From equilibrium the cut portion would contain a direct shear force $F$ and a torsion $T = FD/2$.
- Max stress in wire may be computed by

$$\tau_{\text{max}} = \frac{Tr}{J} + \frac{F}{A}$$

- $T_{\text{max}} = \tau$, $T = FD/2$, $r = d/2$, $J = \pi d^4/32$, & $A = \pi d^2/4$ gives

$$\tau = \frac{8FD}{\pi d^3} + \frac{4F}{\pi d^2}$$

Figure 10–1: (a) Axially loaded helical spring; (b) FBD showing that wire is subjected to a direct shear & torsional shear.
10–1 Stresses in Helical Springs

- *spring index, C*: measure of coil curvature

\[ C = \frac{D}{d} \hspace{10cm} (10-1) \]

- Preferred value of C: 4 to 12

\[ \tau = K_s \frac{8FD}{\pi d^3} \hspace{10cm} (10-2) \]

- \( K_s = \text{shear stress-correction factor} \) and is given by

\[ K_s = \frac{2C + 1}{2C} \hspace{10cm} (10-3) \]
10–2 The Curvature Effect

Equation (10–2) is based on the wire being straight. However, the curvature of the wire increases the stress on the inside of the spring but decreases it only slightly on the outside.

This curvature stress is primarily important in fatigue because the loads are lower and there is no opportunity for localized yielding.

For static loading, these stresses can normally be neglected because of strain-strengthening with the first application of load.
10–2 The Curvature Effect

- Suppose \( K_s \) in Eq. (10–2) is replaced by another \( K \) factor, which corrects for both curvature and direct shear. Then this factor is given by either of:

\[
K_W = \frac{4C - 1}{4C - 4} + \frac{0.615}{C} \tag{10–4}
\]

\[
K_B = \frac{4C + 2}{4C - 3} \tag{10–5}
\]

- Eq. 10-4: Wahl factor, Eq. 10-5: Bergsträsser factor
10–2 The Curvature Effect

- The curvature correction factor can now be obtained by canceling out the effect of the direct shear.
- Thus, using Eq. (10–5) with Eq. (10–3), the curvature correction factor is found to be

\[
K_c = \frac{K_B}{K_s} = \frac{2C(4C + 2)}{(4C - 3)(2C + 1)}
\] 

(10–6)

- In this course, to predict the largest shear stress, use

\[
\tau = K_B \frac{8FD}{\pi d^3}
\] 

(10–7)
10–3 Deflection of Helical Springs

- Using Castigliano’s theorem, total strain energy for a helical spring is composed of a torsional component and a shear component.
- From Eqs. (4–18) and (4–20), p. 162, the strain energy is
  \[ U = \frac{T^2 l}{2GJ} + \frac{F^2 l}{2AG} \]  
  (a)
- Substituting \( T = FD/2, I = \pi DN, J = \pi d^4/32, \) and \( A = \pi d^2/4 \) results in
  \[ U = \frac{4F^2 D^3 N}{d^4 G} + \frac{2F^2 DN}{d^2 G} \]
- \( N = N_a = \) number of active coils

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10–3 Deflection of Helical Springs

\[
y = \frac{\partial U}{\partial F} = \frac{8FD^3N}{d^4G} + \frac{4FDN}{d^2G}
\]

\[
y = \frac{8FD^3N}{d^4G} \left(1 + \frac{1}{2C^2}\right) = \frac{8FD^3N}{d^4G}
\]

\[
k = \frac{d^4G}{8D^3N}
\] (10–8)

(10–9)
10–4 Compression Springs

Figure 10–2: Types of ends for compression springs
## 10–4 Compression Springs

**Table 10–1:** Formulas for Dimensional Characteristics of Compression-Springs. \(N_a = \text{Number of Active Coils}\)

<table>
<thead>
<tr>
<th>Term</th>
<th>Plain</th>
<th>Plain and Ground</th>
<th>Squared or Closed</th>
<th>Squared and Ground</th>
</tr>
</thead>
<tbody>
<tr>
<td>End coils, (N_e)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Total coils, (N_t)</td>
<td>(N_a)</td>
<td>(N_a + 1)</td>
<td>(N_a + 2)</td>
<td>(N_a + 2)</td>
</tr>
<tr>
<td>Free length, (L_0)</td>
<td>(pN_a + d)</td>
<td>(p(N_a + 1))</td>
<td>(pN_a + 3d)</td>
<td>(pN_a + 2d)</td>
</tr>
<tr>
<td>Solid length, (L_s)</td>
<td>(d(N_t + 1))</td>
<td>(dN_t)</td>
<td>(d(N_t + 1))</td>
<td>(dN_t)</td>
</tr>
<tr>
<td>Pitch, (p)</td>
<td>((L_0 - d)/N_a)</td>
<td>(L_0/(N_a + 1))</td>
<td>((L_0 - 3d)/N_a)</td>
<td>((L_0 - 2d)/N_a)</td>
</tr>
</tbody>
</table>

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10–4 Compression Springs

Set removal or presetting

- A process used in manufacture of compression springs to induce useful residual stresses.
- It is done by making spring longer than needed and then compressing it to its solid height. This operation sets the spring to required final free length.
- Since torsional yield strength has been exceeded, induces residual stresses opposite in direction to those induced in service.
- Springs to be preset should be designed so that 10 to 30% of initial free length is removed during the operation.
- Set removal increases strength of spring and so is especially useful when spring is used for energy-storage purposes.
- Set removal should not be used when springs are subject to fatigue.
10–5 Stability

Compression coil springs may buckle when the deflection becomes too large. Critical deflection is given by:

\[ y_{cr} = L_0 C_1' \left[ 1 - \left(1 - \frac{C_2'}{\lambda_{eff}^2}\right)^{1/2} \right] \]  \hspace{1cm} (10–10)

\[ \lambda_{eff} = \text{effective slenderness ratio} \text{ and is given by} \]

\[ \lambda_{eff} = \frac{\alpha L_0}{D} \]  \hspace{1cm} (10–11)

\[ C_1' = \frac{E}{2(E - G)} \]

\[ C_2' = \frac{2\pi^2(E - G)}{2G + E} \]
# 10–5 Stability

## Table 10–2: End-Condition Constants $\alpha$ for Helical Compression Springs*

<table>
<thead>
<tr>
<th>End Condition</th>
<th>Constant $\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spring supported between flat parallel surfaces (fixed ends)</td>
<td>0.5</td>
</tr>
<tr>
<td>One end supported by flat surface perpendicular to spring axis (fixed); other end pivoted (hinged)</td>
<td>0.707</td>
</tr>
<tr>
<td>Both ends pivoted (hinged)</td>
<td>1</td>
</tr>
<tr>
<td>One end clamped; other end free</td>
<td>2</td>
</tr>
</tbody>
</table>

*Ends supported by flat surfaces must be squared and ground.*

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10–5 Stability

The condition for absolute stability

\[ L_0 < \frac{\pi D}{\alpha} \left[ \frac{2(E - G)}{2G + E} \right]^{1/2} \]  

For steels, this turns out to be

\[ L_0 < 2.63 \frac{D}{\alpha} \]  

For squared and ground ends \( \alpha = 0.5 \) and \( L_0 < 5.26D \).
10–6 Spring Materials

- In general, pre-hardened wire should not be used if \( D/d < 4 \) or if \( d > 1/4 \) in
- Table 10–3: High-Carbon and Alloy steels
- UNS steels listed in Appendix A should be used in designing hot-worked, heavy-coil springs, as well as flat springs, leaf springs, and torsion bars.
## 10–6 Spring Materials

**Table 10–4:** Constants $A$ and $m$ of $S_{ut} = A/d^m$ for Estimating Minimum Tensile Strength of Common Spring Wires

<table>
<thead>
<tr>
<th>Material</th>
<th>ASTM No.</th>
<th>Exponent $m$</th>
<th>Diameter, in</th>
<th>$A$, kpsi · in$^m$</th>
<th>Diameter, mm</th>
<th>$A$, MPa · mm$^m$</th>
<th>Relative Cost of Wire</th>
</tr>
</thead>
<tbody>
<tr>
<td>Music wire*</td>
<td>A228</td>
<td>0.145</td>
<td>0.004–0.256</td>
<td>201</td>
<td>0.10–6.5</td>
<td>2211</td>
<td>2.6</td>
</tr>
<tr>
<td>OQ&amp;T wire†</td>
<td>A229</td>
<td>0.187</td>
<td>0.020–0.500</td>
<td>147</td>
<td>0.5–12.7</td>
<td>1855</td>
<td>1.3</td>
</tr>
<tr>
<td>Hard-drawn wire‡</td>
<td>A227</td>
<td>0.190</td>
<td>0.028–0.500</td>
<td>140</td>
<td>0.7–12.7</td>
<td>1783</td>
<td>1.0</td>
</tr>
<tr>
<td>Chrome-vanadium wire§</td>
<td>A232</td>
<td>0.168</td>
<td>0.032–0.437</td>
<td>169</td>
<td>0.8–11.1</td>
<td>2005</td>
<td>3.1</td>
</tr>
<tr>
<td>Chrome-silicon wire†</td>
<td>A401</td>
<td>0.108</td>
<td>0.063–0.375</td>
<td>202</td>
<td>1.6–9.5</td>
<td>1974</td>
<td>4.0</td>
</tr>
<tr>
<td>302 Stainless wire#</td>
<td>A313</td>
<td>0.146–0.263</td>
<td>0.013–0.20</td>
<td>169</td>
<td>0.3–2.5</td>
<td>1867</td>
<td>7.6–11</td>
</tr>
<tr>
<td>302 Stainless wire#</td>
<td>A313</td>
<td>0.263–0.478</td>
<td>0.10–0.40</td>
<td>128</td>
<td>2.5–5</td>
<td>2065</td>
<td></td>
</tr>
<tr>
<td>Phosphor-bronze wire**</td>
<td>B159</td>
<td>0</td>
<td>0.004–0.022</td>
<td>145</td>
<td>0.1–0.6</td>
<td>1000</td>
<td>8.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.028</td>
<td>0.022–0.075</td>
<td>121</td>
<td>0.6–2</td>
<td>913</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.064</td>
<td>0.075–0.30</td>
<td>110</td>
<td>2–7.5</td>
<td>932</td>
<td></td>
</tr>
</tbody>
</table>

\[
S_{ut} = \frac{A}{d^m}
\]

\[
0.35S_{ut} \leq S_{sy} \leq 0.52S_{ut}
\]
# 10–6 Spring Materials

Table 10–5: Mechanical Properties of Some Spring Wires

<table>
<thead>
<tr>
<th>Material</th>
<th>Elastic Limit, Percent of $S_{ult}$</th>
<th>Diameter $d$, in</th>
<th>$E$ Mpsi</th>
<th>$G$ GPa</th>
<th>$E$ Mpsi</th>
<th>$G$ GPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Music wire A228</td>
<td>65–75 / 45–60</td>
<td>&lt;0.032</td>
<td>29.5</td>
<td>203.4</td>
<td>12.0</td>
<td>82.7</td>
</tr>
<tr>
<td></td>
<td>0.033–0.063</td>
<td>29.0</td>
<td>200</td>
<td>11.85</td>
<td>81.7</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.064–0.125</td>
<td>28.5</td>
<td>196.5</td>
<td>11.75</td>
<td>81.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>&gt;0.125</td>
<td>28.0</td>
<td>193</td>
<td>11.6</td>
<td>80.0</td>
<td></td>
</tr>
<tr>
<td>HD spring A227</td>
<td>60–70 / 45–55</td>
<td>&lt;0.032</td>
<td>28.8</td>
<td>198.6</td>
<td>11.7</td>
<td>80.7</td>
</tr>
<tr>
<td></td>
<td>0.033–0.063</td>
<td>28.7</td>
<td>197.9</td>
<td>11.6</td>
<td>80.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.064–0.125</td>
<td>28.6</td>
<td>197.2</td>
<td>11.5</td>
<td>79.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>&gt;0.125</td>
<td>28.5</td>
<td>196.5</td>
<td>11.4</td>
<td>78.6</td>
<td></td>
</tr>
<tr>
<td>Oil tempered A239</td>
<td>85–90 / 45–50</td>
<td>28.5</td>
<td>196.5</td>
<td>11.2</td>
<td>77.2</td>
<td></td>
</tr>
<tr>
<td>Valve spring A230</td>
<td>85–90 / 50–60</td>
<td>28.5</td>
<td>196.5</td>
<td>11.2</td>
<td>77.2</td>
<td></td>
</tr>
<tr>
<td>Chrome-vanadium A231</td>
<td>88–93 / 65–75</td>
<td>28.5</td>
<td>196.5</td>
<td>11.2</td>
<td>77.2</td>
<td></td>
</tr>
<tr>
<td>A232</td>
<td>88–93 / 65–75</td>
<td>28.5</td>
<td>196.5</td>
<td>11.2</td>
<td>77.2</td>
<td></td>
</tr>
<tr>
<td>Chrome-silicon A401</td>
<td>85–93 / 65–75</td>
<td>28.5</td>
<td>196.5</td>
<td>11.2</td>
<td>77.2</td>
<td></td>
</tr>
<tr>
<td>Stainless steel</td>
<td></td>
<td>28</td>
<td>193</td>
<td>10</td>
<td>69.0</td>
<td></td>
</tr>
<tr>
<td>A313*</td>
<td>65–75 / 45–55</td>
<td>29.5</td>
<td>208.4</td>
<td>11</td>
<td>75.8</td>
<td></td>
</tr>
<tr>
<td>17-7PH</td>
<td>75–80 / 55–60</td>
<td>29</td>
<td>200</td>
<td>11.2</td>
<td>77.2</td>
<td></td>
</tr>
<tr>
<td>414</td>
<td>65–70 / 42–55</td>
<td>29</td>
<td>200</td>
<td>11.2</td>
<td>77.2</td>
<td></td>
</tr>
<tr>
<td>420</td>
<td>65–75 / 45–55</td>
<td>30</td>
<td>206</td>
<td>11.5</td>
<td>79.3</td>
<td></td>
</tr>
<tr>
<td>431</td>
<td>72–76 / 50–55</td>
<td>15</td>
<td>103.4</td>
<td>6</td>
<td>41.4</td>
<td></td>
</tr>
<tr>
<td>Phosphor-bronze B159</td>
<td>75–80 / 45–50</td>
<td>17</td>
<td>117.2</td>
<td>6.5</td>
<td>44.8</td>
<td></td>
</tr>
<tr>
<td>Beryllium-copper B197</td>
<td>70 / 50</td>
<td>19</td>
<td>131</td>
<td>7.3</td>
<td>50.3</td>
<td></td>
</tr>
<tr>
<td>Inconel alloy X-750</td>
<td>65–70 / 40–45</td>
<td>31</td>
<td>213.7</td>
<td>11.2</td>
<td>77.2</td>
<td></td>
</tr>
</tbody>
</table>
10–6 Spring Materials

**Table 10–6:** Maximum Allowable Torsional Stresses for Helical Compression Springs in Static Applications

<table>
<thead>
<tr>
<th>Material</th>
<th>Maximum Percent of Tensile Strength Before Set Removed (includes $K_W$ or $K_B$)</th>
<th>Maximum Percent of Tensile Strength After Set Removed (includes $K_s$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Music wire and cold-drawn carbon steel</td>
<td>45</td>
<td>60–70</td>
</tr>
<tr>
<td>Hardened and tempered carbon and low-alloy steel</td>
<td>50</td>
<td>65–75</td>
</tr>
<tr>
<td>Austenitic stainless steels</td>
<td>35</td>
<td>55–65</td>
</tr>
<tr>
<td>Nonferrous alloys</td>
<td>35</td>
<td>55–65</td>
</tr>
</tbody>
</table>

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Example 10–1

A helical compression spring is made of no. 16 music wire. The outside coil diameter of the spring is 7/16 in. The ends are squared and there are 12.5 total turns.

a. Estimate the torsional yield strength of the wire.
b. Estimate the static load corresponding to the yield strength.
c. Estimate the scale of the spring.
d. Estimate the deflection that would be caused by load in part (b).
e. Estimate the solid length of the spring.
f. What length should the spring be to ensure that when it is compressed solid and then released, there will be no permanent change in the free length?
g. Given the length found in part (f), is buckling a possibility?
h. What is the pitch of the body coil?

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10–7 Helical Compression Spring
Design for Static Service

1. Spring index: $4 \leq C \leq 12$
   - Lower indexes being more difficult to form because of danger of surface cracking
   - Higher indexes tending to tangle often enough to require individual packing

2. Recommended range of active turns $3 \leq N_a \leq 15$
   - Designer confines spring’s operating point to central 75% of curve between no load, $F = 0$, and closure, $F = F_s$
   - Max operating force should be limited to $F_{\text{max}} \leq 0.875 F_s$

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10–7 Helical Compression Spring Design for Static Service

3. Defining the fractional overrun to closure as $\xi$, where

$$ F_s = (1 + \xi) F_{\text{max}} $$

$$ F_s = (1 + \xi) F_{\text{max}} = (1 + \xi) \left(\frac{7}{8}\right) F_s $$

(10–17)

$$ \xi \geq 0.15 $$

4. $n_s =$ factor of safety at closure (solid height)

$$ n_s \geq 1.2 $$

(10–21)

5. **Figure of merit (fom): cost of wire from which spring is wound.**

$$ \text{fom} = -(\text{relative material cost}) \frac{\gamma \pi^2 d^2 N_t D}{4} $$

(10–22)
10–7 Helical Compression Spring Design for Static Service

Spring Design Software

- MatLab
- Mathematica
- Spread Sheet (MS Excel)
- Some Major Spring Manufactures offer free software consultation for spring selection

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10–7 Helical Compression Spring
Design for Static Service

*Design Strategy* (Fig. 10–3)

- With hard-drawn steel wire the first choice (relative material cost is 1.0).
- Choose a wire size \(d\).
- With all decisions made, generate a column of parameters: \(d, D, C, OD\) or \(ID, N_a, L_s, L_0, (L_0)_{cr}, n_s,\) and \(fom\).
- By incrementing wire sizes available, we can scan table of parameters and apply design recommendations by inspection.
- After wire sizes are eliminated, choose spring design with highest \(fom\).
10–7 Helical Compression Spring
Design for Static Service

Choose \( d \)

- Over-a-rod
  - As-wound or set
    - \( D = d_{rod} + d + \text{allow} \)
    - \( S_{sy} = \text{const}(A)/d^m \)
  - As-wound
    - \( S_{sy} = 0.65A/d^m \)
- Free
  - Set removed
    - \( D = 0.65 \pi d^3 / (8n_s(1 + \xi)F_{max}) \)
  - As-wound or set
    - \( D = C_d \)
- In-a-hole
  - \( D = d_{hole} - d - \text{allow} \)

\( C = \frac{2\alpha - \beta}{4\beta} + \sqrt{\left(\frac{2\alpha - \beta}{4\beta}\right)^2 - \frac{3\alpha}{4\beta}} \)

\( \alpha = \frac{S_{sy}}{n_s} \)

\( \beta = \frac{8(1 + \xi)F_{max}}{\pi d^2} \)

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10–7 Helical Compression Spring

Design for Static Service

\[ C = \frac{D}{d} \]

\[ K_B = \frac{(4C + 2)}{(4C - 3)} \]

\[ \tau_s = K_B 8(1 + \xi)F_{\text{max}}D / (\pi d^3) \]

\[ n_s = \frac{S_{sy}}{\tau_s} \]

\[ \text{OD} = D + d \]

\[ \text{ID} = D - d \]

\[ N_a = Gd^4 y_{\text{max}} / (8D^3 F_{\text{max}}) \]

\[ N_i: \text{Table 10–1} \]

\[ L_s: \text{Table 10–1} \]

\[ L_O: \text{Table 10–1} \]

\[ (L_O)_{ct} = 2.63D/\alpha \]

\[ \text{fom} = -(\text{rel. cost}) \gamma \pi^2 d^2 N_i D / 4 \]

Print or display: \( d, D, C, \text{OD}, \text{ID}, N_a, N_i, L_s, L_O, (L_O)_{ct}, n_s \), from Build a table, conduct design assessment by inspection Eliminate infeasible designs by showing active constraints Choose among satisfactory designs using the figure of merit \(^\dagger\) const is found from Table 10–6.

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Example 10–2

A music wire helical compression spring is needed to support a 20-lbf load after being compressed 2 in. Because of assembly considerations the solid height cannot exceed 1 in and the free length cannot be more than 4 in. Design the spring.
Example 10–3

Design a compression spring with plain ends using hard-drawn wire. The deflection is to be 2.25 in when the force is 18 lbf and to close solid when the force is 24 lbf. Upon closure, use a design factor of 1.2 guarding against yielding. Select the smallest gauge W&M (Washburn & Moen) wire.
10–8 Critical Frequency of Helical Springs

- **spring surge**: If one end of a compression spring is held against a flat surface and the other end is disturbed, a compression wave is created that travels back and forth from one end to the other exactly like the swimming-pool wave.

- When helical springs are used in applications requiring a rapid reciprocating motion, the designer must be certain that the physical dimensions of the spring are not such as to create a natural vibratory frequency close to the frequency of the applied force;
  - otherwise, **resonance** may occur, resulting in damaging stresses, since the internal damping of spring materials is quite low.

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10–8 Critical Frequency of Helical Springs

The governing equation for translational vibration of a spring is the wave equation

\[
\frac{\partial^2 u}{\partial x^2} = \frac{W}{kgl^2} \frac{\partial^2 u}{\partial t^2}
\]

- \(k\) = spring rate
- \(g\) = acceleration due to gravity
- \(l\) = length of spring
- \(W\) = weight of spring
- \(x\) = coordinate along length of spring
- \(u\) = motion of any particle at distance \(x\)
10–8 Critical Frequency of Helical Springs

- The solution to this equation is harmonic and depends on given physical properties as well as the end conditions of the spring. The harmonic, natural, frequencies for a spring placed between two flat and parallel plates, in radians per second, are

\[ \omega = m\pi \sqrt{\frac{kg}{W}} \quad m = 1, 2, 3, \ldots \]

- Fundamental frequency in hertz,

\[ f = \frac{1}{2} \sqrt{\frac{kg}{W}} \quad \text{(10–25)} \]

- Where the spring has one end against a flat plate and the other end free, Wolford and Smith show that the frequency is

\[ f = \frac{1}{4} \sqrt{\frac{kg}{W}} \quad \text{(10–26)} \]
**10–8 Critical Frequency of Helical Springs**

- Eq. (10–25) applies when one end is against a flat plate and the other end is driven with a sine-wave motion.
  \[ f = \frac{1}{2} \sqrt{\frac{kg}{W}} \]  
  \[ (10–25) \]

- The weight of the active part of a helical spring is
  \[ W = AL\gamma = \frac{\pi d^2}{4} (\pi DN_a)(\gamma) = \frac{\pi^2 d^2 DN_a \gamma}{4} \]  
  \[ (10–27) \]

- Fundamental critical frequency > 15 to 20 times frequency of force or motion of the spring in order to avoid resonance with the harmonics.

- If frequency is not high enough, spring should be redesigned to increase \( k \) or decrease \( W \).
10–9 Fatigue Loading of Helical Compression Springs

- Shot peening is used to improve fatigue strength of dynamically loaded springs. It can increase torsional fatigue strength by 20% or more.

- Zimmerli: size, material, and tensile strength have no effect on endurance limits (infinite life) of spring steels in sizes under 3/8 in (10 mm).

- Endurance limits tend to level out at high tensile strengths (Fig. 6–17)

- Unpeened springs were tested from a min torsional stress of 20 kpsi to a max of 90 kpsi and peened springs in the range 20 kpsi to 135 kpsi.
10–9 Fatigue Loading of Helical Compression Springs

\[ S'_{e} = \begin{cases} 
0.5S_{ut} & S_{ut} \leq 200 \text{ kpsi (1400 MPa)} \\
100 \text{ kpsi} & S_{ut} > 200 \text{ kpsi} \\
700 \text{ MPa} & S_{ut} > 1400 \text{ MPa} 
\end{cases} \]

Figure 6-17
10–9 Fatigue Loading of Helical Compression Springs

- Endurance strength components for infinite life:
  - Unpeened: \( S_{sa} = 35 \text{ kpsi} (241 \text{ MPa}) \) \( S_{sm} = 55 \text{ kpsi} (379 \text{ MPa}) \)
  - Peened: \( S_{sa} = 57.5 \text{ kpsi} (398 \text{ MPa}) \) \( S_{sm} = 77.5 \text{ kpsi} (534 \text{ MPa}) \)

- For example, given an unpeened spring with \( S_{su} = 211.5 \text{ kpsi} \), the Gerber ordinate intercept for shear, Eq. (6–42), p. 306,

\[
S_{se} = \frac{S_{sa}}{1 - \left( \frac{S_{sm}}{S_{su}} \right)^2} = \frac{35}{1 - \left( \frac{55}{211.5} \right)^2} = 37.5 \text{ kpsi}
\]

- For Goodman failure criterion, the intercept would be 47.3 kpsi. Each possible wire size would change these numbers, since \( S_{su} \) would change.
Commonly Used Failure Criteria

- Gerber passes through the data
- ASME-elliptic passes through data and incorporates rough yielding check

- Modified Goodman is linear, so simple to use for design. It is more conservative than Gerber.
- Soderberg provides a very conservative single check of both fatigue and yielding.

- Langer line represents standard yield check.
- It is equivalent to comparing max stress to yield strength.

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Equations for Commonly Used Failure Criteria

- Intersecting a constant slope load line with each failure criteria produces design equations
- $n$ is the design factor or factor of safety for infinite fatigue life

\[
\text{Soderberg} \quad \frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} = \frac{1}{n} \quad (6-45)
\]

\[
\text{mod-Goodman} \quad \frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}} = \frac{1}{n} \quad (6-46)
\]

\[
\text{Gerber} \quad \frac{n\sigma_a}{S_e} + \left( \frac{n\sigma_m}{S_{ut}} \right)^2 = 1 \quad (6-47)
\]

\[
\text{ASME-elliptic} \quad \left( \frac{n\sigma_a}{S_e} \right)^2 + \left( \frac{n\sigma_m}{S_y} \right)^2 = 1 \quad (6-48)
\]
### Summarizing Table for Gerber

#### Table 6–7

<table>
<thead>
<tr>
<th>Intersecting Equations</th>
<th>Intersection Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{S_a}{S_e} + \left( \frac{S_m}{S_{ut}} \right)^2 = 1 )</td>
<td>( S_a = \frac{r^2 S_{ut}^2}{2S_e} \left[ -1 + \sqrt{1 + \left( \frac{2S_e}{r S_{ut}} \right)^2} \right] )</td>
</tr>
<tr>
<td>Load line ( r = \frac{S_a}{S_m} )</td>
<td>( S_m = \frac{S_a}{r} )</td>
</tr>
<tr>
<td>( \frac{S_a}{S_y} + \frac{S_m}{S_y} = 1 )</td>
<td>( S_a = \frac{r S_y}{1 + r} )</td>
</tr>
<tr>
<td>Load line ( r = \frac{S_a}{S_m} )</td>
<td>( S_m = \frac{S_y}{1 + r} )</td>
</tr>
<tr>
<td>( \frac{S_a}{S_e} + \left( \frac{S_m}{S_{ut}} \right)^2 = 1 )</td>
<td>( S_m = \frac{S_{ut}^2}{2S_e} \left[ 1 - \sqrt{1 + \left( \frac{2S_e}{S_{ut}} \right)^2 \left( 1 - \frac{S_y}{S_e} \right)} \right] )</td>
</tr>
<tr>
<td>( \frac{S_a}{S_y} + \frac{S_m}{S_y} = 1 )</td>
<td>( S_a = S_y - S_m, \ r_{crit} = \frac{S_a}{S_m} )</td>
</tr>
</tbody>
</table>

Fatigue factor of safety

\[
n_f = \frac{1}{2} \left( \frac{S_{ut}}{\sigma_m} \right)^2 \frac{\sigma_a}{S_e} \left[ -1 + \sqrt{1 + \left( \frac{2\sigma_m S_e}{S_{ut} \sigma_a} \right)^2} \right], \quad \sigma_m > 0
\]

Dr. Mohammad Suliman Abuhaiba, PE
10–9 Fatigue Loading of Helical Compression Springs

Sines Failure Criterion in Torsional Fatigue

- For polished, notch-free, cylindrical specimens subjected to torsional shear stress,
  - Max alternating stress that may be imposed without causing failure is constant and independent of mean stress in the cycle
  - Max stress range does not equal or exceed torsional yield strength of the metal.

- Torsional modulus of rupture $S_{su}$ is given by:

\[
S_{su} = 0.67S_{ut} \\
F_a = \frac{F_{\text{max}} - F_{\text{min}}}{2} \\
F_m = \frac{F_{\text{max}} + F_{\text{min}}}{2}
\] 

(10–30)

(10–31a)

(10–31b)
10–9 Fatigue Loading of Helical Compression Springs

\[ \tau_a = K_B \frac{8 F_a D}{\pi d^3} \]  

\[ \tau_m = K_B \frac{8 F_m D}{\pi d^3} \]  

(10–32)  

(10–33)
Example 10–4

An as-wound helical compression spring, made of music wire, has a wire size of 0.092 in, an outside coil diameter of 9/16 in, a free length of 4 3/8 in, 21 active coils, and both ends squared and ground. The spring is unpeened. This spring is to be assembled with a preload of 5 lbf and will operate with a maximum load of 35 lbf during use.

a. Estimate factor of safety guarding against fatigue failure using a torsional Gerber fatigue failure criterion with Zimmerli data.

b. Repeat part (a) using Sines torsional fatigue criterion (steady stress component has no effect), with Zimmerli data.

c. Repeat using (a) torsional Goodman failure criterion with Zimmerli data.

d. Estimate the critical frequency of the spring.
Example 10–5

A music wire helical compression spring with infinite life is needed to resist a dynamic load that varies from 5 to 20 lbf at 5 Hz while the end deflection varies from 1/2 to 2 in. Because of assembly considerations, the solid height cannot exceed 1 in and the free length cannot be more than 4 in. The spring maker has the following wire sizes in stock: 0.069, 0.071, 0.080, 0.085, 0.090, 0.095, 0.105, and 0.112 in.

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10–11 Extension Springs

Figure 10–5: Types of ends used on extension springs. (Courtesy of Associated Spring.)
10–11 Extension Springs

Figure 10–6: Ends for extension springs. (a) Usual design; stress at A is due to combined axial force and bending moment. (b) Side view of part a; stress is mostly torsion at B.

- Radius $r_1$ is in the plane of the end coil for curved beam bending stress.
- Radius $r_2$ is at a right angle to the end coil for torsional shear stress.
10–11 Extension Springs

Figure 10–6: Ends for extension springs. (c) Improved design; stress at A is due to combined axial force and bending moment. (d) Side view of part c; stress at B is mostly torsion.

- Radius $r_1$ is in the plane of the end coil for curved beam bending stress.
- Radius $r_2$ is at a right angle to the end coil for torsional shear stress.
10–11 Extension Springs

- Max tensile stress at A, due to bending & axial loading, is given by

\[ \sigma_A = F \left[ (K)_A \frac{16D}{\pi d^3} + \frac{4}{\pi d^2} \right] \]  \hfill (10–34)

- \((K)_A\) is a bending stress-correction factor for curvature, given by

\[ (K)_A = \frac{4C_1^2 - C_1 - 1}{4C_1(C_1 - 1)} \quad \text{and} \quad C_1 = \frac{2r_1}{d} \]  \hfill (10–35)

- Max torsional stress at point B is given by

\[ \tau_B = (K)_B \frac{8FD}{\pi d^3} \]  \hfill (10–36)

\[ (K)_B = \frac{4C_2 - 1}{4C_2 - 4} \quad \text{and} \quad C_2 = \frac{2r_2}{d} \]  \hfill (10–37)
10–11 Extension Springs

Figure 10–7: (a) Geometry of the force $F$ and extension $y$ curve of an extension spring; (b) geometry of the extension spring

\[ F = F_i + ky \]

\[ L_0 = 2(D - d) + (N_b + 1)d = (2C - 1 + N_b)d \]

$N_b$ is the number of body coils

the equivalent number of active helical turns $N_a$ for use in Eq. (10–9) is

\[ N_a = N_b + \frac{G}{E} \]
Figure 10–7: (c) torsional stresses due to initial tension as a function of spring index $C$ in helical extension springs.

The preferred range of Uncorrected torsional stress $\tau_i$ as

$$
\tau_i = \frac{33 \, 500}{\exp(0.105C)} \pm 1000 \left(4 - \frac{C - 3}{6.5}\right) \text{ psi}
$$

(10–41)
### 10–11 Extension Springs

**Table 10–7:** Max Allowable Stresses ($K_W$ or $K_B$ corrected) for Helical Extension Springs in Static Applications

<table>
<thead>
<tr>
<th>Materials</th>
<th>Percent of Tensile Strength</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>In Torsion</td>
</tr>
<tr>
<td>Patented, cold-drawn or hardened and tempered carbon and low-alloy steels</td>
<td>45–50</td>
</tr>
<tr>
<td>Austenitic stainless steel and nonferrous alloys</td>
<td>35</td>
</tr>
</tbody>
</table>

This information is based on the following conditions: set not removed and low temperature heat treatment applied. For springs that require high initial tension, use the same percent of tensile strength as for end.
Example 10–6

A hard-drawn steel wire extension spring has a wire diameter of 0.035 in, an outside coil diameter of 0.248 in, hook radii of $r_1 = 0.106$ in and $r_2 = 0.089$ in, and an initial tension of 1.19 lbf. The number of body turns is 12.17. From the given information:

a. Determine the physical parameters of the spring.
b. Check the initial preload stress conditions.
c. Find factors of safety under a static 5.25-lbf load.
Example 10–7

The helical coil extension spring of Ex. 10–6 is subjected to a dynamic loading from 1.5 to 5 lbf. Estimate the factors of safety using the Gerber failure criterion for:

a. coil fatigue
b. coil yielding
c. end-hook bending fatigue at point A of Fig. 10–6a
d. end-hook torsional fatigue at point B of Fig. 10–6b
The end hooks are usually the weakest part, with bending usually controlling.

A fatigue failure separates the extension spring under load.

Flying fragments, lost load, and machine shutdown are threats to personal safety as well as machine function.

For these reasons higher design factors are used in extension-spring.
10–11 Extension Springs

- In Ex. 10–7 we estimated the endurance limit for the hook in bending using Zimmerli data, which are based on torsion in compression springs and the distortion theory.
- An alternative method is to use Table 10–8, which is based on a stress-ratio of $R = \frac{\tau_{\text{min}}}{\tau_{\text{max}}} = 0$. For this case, $\tau_a = \tau_m = \frac{\tau_{\text{max}}}{2}$.

<table>
<thead>
<tr>
<th>Number of Cycles</th>
<th>Percent of Tensile Strength In Torsion</th>
<th>Percent of Tensile Strength In Bending</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Body</td>
<td>End</td>
</tr>
<tr>
<td>$10^5$</td>
<td>36</td>
<td>34</td>
</tr>
<tr>
<td>$10^6$</td>
<td>33</td>
<td>30</td>
</tr>
<tr>
<td>$10^7$</td>
<td>30</td>
<td>28</td>
</tr>
</tbody>
</table>

This information is based on the following conditions: not shot-peened, no surging and ambient environment with a low temperature heat treatment applied. Stress ratio = 0.

Dr. Mohammad Suliman Abuhaiba, PE
10–11 Extension Springs

- Label the strength values of Table 10–8 as $S_r$ for bending or $S_{sr}$ for torsion.
- Then for torsion, for example, $S_{sa} = S_{sm} = S_{sr}/2$ and the Gerber ordinate intercept, given by Eq. (6–42) for shear, is

$$S_{se} = \frac{S_{sa}}{1 - (S_{sm}/S_{su})^2} = \frac{S_{sr}/2}{1 - \left(\frac{S_{sr}/2}{S_{su}}\right)^2} \quad (10–42)$$

- In Ex. 10–7 an estimate for the bending endurance limit from Table 10–8 would be

$$S_r = 0.45S_{ut} = 0.45(264.7) = 119.1 \text{ kpsi}$$

$$S_e = \frac{S_r/2}{1 - [S_r/(2S_{ut})]^2} = \frac{119.1/2}{1 - \left(\frac{119.1/2}{264.7}\right)^2} = 62.7 \text{ kpsi}$$

- Using this in place of 67.1 kpsi in Ex. 10–7 results in $(nf)A = 1.03$, a reduction of 5%
10–12 Helical Coil Torsion Springs

Usually close-wound with negligible initial tension

- Short hook ends
- Hinge ends
- Straight offset
- Straight torsion
- Special ends
- Double torsion

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10–12 Helical Coil Torsion Springs

- The wire in a torsion spring is in bending
- As the applied torque increases, the inside diameter of the coil decreases.
10–12 Helical Coil Torsion Springs

Describing the End Location

Table 10–9: End Position Tolerances for Helical Coil Torsion Springs (for $D/d$ Ratios up to and Including 16)

<table>
<thead>
<tr>
<th>Total Coils</th>
<th>Tolerance: ± Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up to 3</td>
<td>8</td>
</tr>
<tr>
<td>Over 3–10</td>
<td>10</td>
</tr>
<tr>
<td>Over 10–20</td>
<td>15</td>
</tr>
<tr>
<td>Over 20–30</td>
<td>20</td>
</tr>
<tr>
<td>Over 30</td>
<td>25</td>
</tr>
</tbody>
</table>
10–12 Helical Coil Torsion Springs

Describing the End Location

The initial unloaded location of one end with respect to the other is in terms of an angle $\beta$ defining the partial turn present in coil body as $N_p = \frac{\beta}{360^\circ}$, Fig. 10–9.

- # of body turns $N_b = \#$ of turns in the free spring body by count
- Body-turn count is related to initial position angle $\beta$ by

$$N_b = \text{integer} + \frac{\beta}{360^\circ} = \text{integer} + N_p$$

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10–12 Helical Coil Torsion Springs

**Bending Stress**

- A torsion spring has bending induced in the coils, rather than torsion.
- Residual stresses built in during winding are in the same direction but of opposite sign to the working stresses that occur during use.
- The strain-strengthening locks in residual stresses opposing working stresses *provided* the load is always applied in the winding sense.

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Bending Stress

Bending stress can be obtained from curved-beam theory expressed in the form

$$\sigma = K \frac{Mc}{I}$$

- $K = \text{stress-correction factor}$
- Value of $K$ depends on:
  1. shape of wire
  2. cross section
  3. whether stress sought is at inner or outer fiber
**10–12 Helical Coil Torsion Springs**

**Bending Stress**

- Wahl analytically determined values of $K$ to be, for round wire,

$$K_i = \frac{4C^2 - C - 1}{4C(C - 1)} \quad K_o = \frac{4C^2 + C - 1}{4C(C + 1)} \quad (10–43)$$

- $K_o$ is always less than unity
- Use $K_i$ to estimate the stresses
- $M = Fr$ and section modulus $l/c = d^3/32$

$$\sigma = K_i \frac{32Fr}{\pi d^3} \quad (10–44)$$
10–12 Helical Coil Torsion Springs

Deflection and Spring Rate

- If a term contains revolution units the term will be expressed with a prime sign.
- Spring rate \( k' \) units = torque/revolution (\( \text{lbf} \cdot \text{in}/\text{rev} \) or \( \text{N} \cdot \text{mm}/\text{rev} \))
- Spring rate, if linear, can be expressed as

\[
k' = \frac{M_1}{\theta'_1} = \frac{M_2}{\theta'_2} = \frac{M_2 - M_1}{\theta'_2 - \theta'_1}
\]  \hspace{1cm} (10–45)

- Total angular deflection in radians:

\[
\theta_t = \frac{64MDN_b}{d^4E} + \frac{64Ml_1}{3\pi d^4E} + \frac{64Ml_2}{3\pi d^4E} = \frac{64MD}{d^4E} \left( N_b + \frac{l_1 + l_2}{3\pi D} \right)
\]  \hspace{1cm} (10–47)

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10–12 Helical Coil Torsion Springs

Deflection and Spring Rate

- Equivalent number of active turns $N_a$ is expressed as

$$N_a = N_b + \frac{l_1 + l_2}{3\pi D}$$

$$k = \frac{F r}{\theta_t} = \frac{M}{\theta_t} = \frac{d^4 E}{64 D N_a}$$

$$k' = \frac{d^4 E}{10.8 D N_a}$$

$$\theta'_t = \frac{10.8 M D}{d^4 E} \left( N_b + \frac{l_1 + l_2}{3\pi D} \right)$$
**10–12 Helical Coil Torsion Springs**

**Deflection and Spring Rate**

- When load is applied to a torsion spring, spring winds up, causing a decrease in inside diameter of coil body.
- The helix diameter of the coil $D'$ becomes

$$D' = \frac{N_b D}{N_b + \theta'_c} \quad (10-53)$$

- $\theta'_c$ is angular deflection of the body of coil in number of turns,

$$\theta'_c = \frac{10.8 M D N_b}{d^4 E} \quad (10-54)$$
10–12 Helical Coil Torsion Springs

Deflection and Spring Rate

Diametral clearance between body coil & pin of diameter $D_p$ equal to

$$\Delta = D' - d - D_p = \frac{N_b D}{N_b + \theta'_c} - d - D_p$$  (10–55)
10–12 Helical Coil Torsion Springs

**Static Strength**

First column entries in Table 10–6 can be divided by 0.577 (from distortion-energy theory) to give

\[
S_y = \begin{cases} 
0.78S_{ut} & \text{Music wire and cold-drawn carbon steels} \\
0.87S_{ut} & \text{OQ&T carbon and low-alloy steels} \\
0.61S_{ut} & \text{Austenitic stainless steel and nonferrous alloys}
\end{cases}
\]  

(10–57)
10–12 Helical Coil Torsion Springs

**Fatigue Strength**

- we will use the repeated bending stress ($R = 0$) values provided by Associated Spring in Table 10–10.

- **Table 10–10:** Max Recommended Bending Stresses ($KB$ Corrected) for Helical Torsion Springs in Cyclic Applications as Percent of $S_{ut}$ **Source:** Courtesy of Associated Spring.

<table>
<thead>
<tr>
<th>Fatigue Life, Cycles</th>
<th>ASTM A228 and Type 302 Stainless Steel</th>
<th>ASTM A230 and A232</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Not Shot-Peened</td>
<td>Shot-Peened*</td>
</tr>
<tr>
<td>$10^5$</td>
<td>53</td>
<td>62</td>
</tr>
<tr>
<td>$10^6$</td>
<td>50</td>
<td>60</td>
</tr>
</tbody>
</table>

This information is based on the following conditions: no surging, springs are in the “as-stress-relieved” condition.

*Not always possible.

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10–12 Helical Coil Torsion Springs

**Fatigue Strength**

- As in Eq. (10–40) we will use the Gerber fatigue-failure criterion incorporating the Associated Spring $R = 0$ fatigue strength $S_r$:

  $$S_e = \frac{S_r / 2}{1 - \left(\frac{S_r / 2}{S_{ut}}\right)^2}$$  \hspace{1cm} (10–58)

- Value of $S_r$ (and $S_e$) has been corrected for size, surface condition, and type of loading, but not for temperature or miscellaneous effects.
10–12 Helical Coil Torsion Springs

**Fatigue Strength**

- Gerber fatigue criterion is now defined. The strength amplitude component is given by Table 6–7, p. 307,

\[
S_a = \frac{r^2 S_{ut}^2}{2 S_e} \left[ -1 + \sqrt{1 + \left( \frac{2 S_e}{r S_{ut}} \right)^2} \right]
\]  
(10–59)

- Slope of load line is \( r = \frac{M_a}{M_m} \). The load line is radial through origin of designer’s fatigue diagram.

- Factor of safety guarding against fatigue failure is

\[
n_f = \frac{S_a}{\sigma_a}
\]  
(10–60)

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Example 10–8

A stock spring is made from 0.072-in-diameter music wire & has 4.25 body turns with straight torsion ends. It works over a pin of 0.400 in diameter. The coil outside diameter is 19/32 in.

a. Find max operating torque & corresponding rotation for static loading.

b. Estimate inside coil diameter & pin diametral clearance when spring is subjected to the torque in part (a).

c. Estimate fatigue factor of safety $n_f$ if applied moment varies between $M_{min} = 1$ to $M_{max} = 5$ lbf . in.

Dr. Mohammad Suliman Abuhaiba, PE
Second Exam

On Tuesday 29/10/2013 at 11:00

Tested Material: Chapter 10
Practice Problems

General Problems

- 3, 7, 20, 23, 25, 28, 36, 39

Design & Programming Problems

- 30 & 44
- 37 & 45