4.7 System Response with Additional Poles

* The formulae describing 0.5%, Ts, & Tp were derived only for a system with two complex poles & no zeros.
* In order to use the formulae for a system with more than two poles or with zeros, the system must be approximated to a second order system that just has two complex dominant poles.

** Effect of an additional pole on second order response.**

If a system has three poles: 2 complex & 1 real pole

\[ (-3\omega n + j\omega n\sqrt{1-3^2}, \omega n) \] then the third real pole must be five times further to the left than the dominant poles so that its effect on the second order response is negligible.

**Skill Assessment Exercise 4.6**

* Determine the validity of a second order approximation for each of these two transfer functions:

(a) \[ G(s) = \frac{700}{(s+15)(s^2+4s+100)} \]

Poles are \(-2 \pm j9.79, -15\) \(\Rightarrow\) approximation is valid since the higher order pole is more than five times further from the real part of the dominant poles.

(b) \[ G(s) = \frac{360}{(s+4)(s^2+25+9)} \]

Poles are \(-1 \pm j2.8, -4\) \(\Rightarrow\) approximation is not valid since the higher order pole is not more than five times further from the real part of the dominant poles.
If a system has two poles & a zero then the closer the zero is to the dominant poles the greater its effect on the transient response. As the zero moves away from the dominant poles, the response approaches that of the two-pole system.

**Zero Pole Cancellation**

**Skill Assessment Exercise 4.7.8**

* Determine the validity of a second-order step response approximation for each transfer function.

\[ G(s) = \frac{185.7}{s(8+7)} \]

\[ C(s) = G(s) \frac{1}{s} = \frac{185.7}{s(8+7)} \]

\[ C(s) = \frac{1}{s} + \frac{0.8942}{s+20} - \frac{1.5918}{s+10} - \frac{0.3023}{s+6.5} \]

**Note:** The pole -6.5 is the closest to the zero -7. So we see the residue of this pole if it is negligible compared to the other residues; then a zero pole cancellation can occur otherwise there will not be cancellation.

* We see that -0.3023 is not negligible compared to the other residues. So there will not be cancellation and the approximation is not valid.
* If we are given a state equation and asked to solve it, we can solve it by two ways:
  1. Laplace technique
  2. Directly by time domain (differential equations)

4.10 Laplace Transform Solution of State Equation

\[ Y(s) = CX(s) + DU(s) \]
\[ X(s) = (SI-A)^{-1}[x_0] + BU(s) \]

Skill Assessment Exercise 4.9

Given the system represented in state space:

\[ \dot{x} = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \]
\[ y = \begin{bmatrix} 1 & 3 \end{bmatrix} x \]
\[ x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

(a) Solve for \( y(t) \) using Laplace transform techniques:

\[ [SI-A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix} = \begin{bmatrix} s & -2 \\ 3 & s+5 \end{bmatrix} \]

\[ (SI-A)^{-1} = \begin{bmatrix} s & -2 \\ 3 & s+5 \end{bmatrix}^{-1} = \frac{1}{s(s+5)+6} \begin{bmatrix} s+5 & 2 \\ 3 & s \end{bmatrix} \]

\[ BU(s) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s+1} = \begin{bmatrix} 0 \\ \frac{1}{s+1} \end{bmatrix} \]
\[
\begin{bmatrix}
X(0) + Bu(s) \\
\end{bmatrix} =
\begin{bmatrix}
2 \\
1
\end{bmatrix} +
\begin{bmatrix}
0 \\
\frac{1}{s+1}
\end{bmatrix} =
\begin{bmatrix}
2 \\
\frac{s+2}{s+1}
\end{bmatrix}
\]

\[X(s) = \frac{1}{s+5s+6}
\]

\[X(s) = \frac{1}{s+5s+6}
\]

\[X(s) = \frac{1}{s+5s+6}
\]

\[y(s) = CX(s)
\]

\[y(s) = \begin{bmatrix} 1 & 3 \end{bmatrix}
\begin{bmatrix}
2s+10 + \frac{4s+4}{s+1} \\
-6 + \frac{8(s+2)}{s+1}
\end{bmatrix}
\]

\[y(s) = \frac{2s + 10 + \frac{2s+4}{s+1}}{s+5s+6} - 18 + \frac{3s(s+2)}{s+1}
\]

\[y(s) = \frac{5s^2 + 2s - 4}{(s+1)(s+5s+6)}
\]

\[y(s) = \frac{-0.5}{s+1} - \frac{12}{s+2} + \frac{17.5}{s+3}
\]

\[y(t) = -0.5 e^{-t} - 12 e^{-2t} + 17.5 e^{-3t}
\]

\[\text{Find eigenvalues & System poles:}
\]

\[\text{S2-A1 = } s^2 + 5s + 6 \quad \text{So poles = -2, -3}
\]
4.11 Time Domain Solution of State Equations

The solution is given by

\[ x(t) = \Phi(t) x(0) + \int_0^t \Phi(t - \tau) B u(\tau) d\tau \]

\[ \Phi(t) = e^{At} \quad \text{State transition matrix} \]

\[ \Phi(t) = e^{(S I - A)^{-1}} \]

Skill Assessment Exercise 4.10

Given the system in S. S'

\[ \begin{align*}
    x &= \begin{bmatrix} 0 & 2 \\ -2 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} \\
    y &= \begin{bmatrix} 2 & 1 \end{bmatrix} x
\end{align*} \]

\[ x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]

(1) Solve for the state transition matrix.

\[ (S I - A)^{-1} = \begin{bmatrix} S + 5 & 2 \\ 2 & S + 5 \end{bmatrix}^{-1} = \frac{1}{S(S + 5) + 4} \begin{bmatrix} 8 + 5 & 2 \\ 2 & 8 + 5 \end{bmatrix} \]

\[ (S I - A)^{-1} = \begin{bmatrix} \frac{S + 5}{S^2 + 5S + 4} & \frac{2}{S^2 + 5S + 4} \\ -\frac{2}{S^2 + 5S + 4} & \frac{S}{S^2 + 5S + 4} \end{bmatrix} \]

\[ \Phi(t) = e^{(S I - A)^{-1}} = \begin{bmatrix} \frac{S + 5}{3} e^{-\frac{t}{3}} + \frac{1}{3} e^{-4t} & \frac{2}{3} e^{-\frac{t}{3}} - \frac{2}{3} e^{-4t} \\ -\frac{2}{3} e^{-\frac{t}{3}} + \frac{2}{3} e^{-4t} & \frac{S}{3} e^{-\frac{t}{3}} + \frac{2}{3} e^{-4t} \end{bmatrix} \]
(b) Solve for the state vector using the convolution integral:

\[ \Phi(t-z) = \begin{bmatrix} \frac{4}{3} e^{-(t-z)} - \frac{1}{3} e^{-(t-z)} & -4(t-z) & -\frac{2}{3} e^{-(t-z)} & -\frac{4}{3} e^{-(t-z)} \\ -\frac{2}{3} e^{-(t-z)} + \frac{2}{3} e^{-(t-z)} & -4(t-z) & -\frac{1}{3} e^{-(t-z)} & -\frac{4}{3} e^{-(t-z)} \end{bmatrix} \]

\[ B u(t) = \begin{bmatrix} 0 & -2t \\ e \end{bmatrix} \]

\[ \Phi(t-z) B u(z) = \begin{bmatrix} \frac{2}{3} e^{-z} e^{-t} - \frac{2}{3} e^{-z} e^{2t} - 4t \\ -\frac{1}{3} e^{-z} e^{-t} + \frac{2}{3} e^{-z} e^{2t} - 4t \end{bmatrix} \]

\[ X(t) = \Phi(t) x(0) + \int_0^t \Phi(t-z) B u(z) d\zeta = \begin{bmatrix} \frac{10}{3} e^{-t} - e^{-2t} - \frac{4}{3} e^{-4t} \\ -\frac{5}{3} e^{-t} + e^{-2t} + \frac{8}{3} e^{-4t} \end{bmatrix} \]

(c) Find the output \( y(t) \)

\[ y(t) = [2, 1] x = 5 e^{-t} - e^{-2t} \]

Good Luck in Midterm Exam 😊