Chapter 4

More Interest Formulas
Uniform Series Compound Interest Formulas

• Why?

Many payments are based on a uniform payment series. e.g. automobile loans, house payments, and many other loans.
The Uniform Payment Series A is ...

The series A is: An end-of-period cash receipt or disbursement in a uniform series, continuing for \( n \) periods, the entire series equivalent to P or F at interest rate \( i \).
“A” payments in terms of “F”

Amount A is invested at the end of each year for four years

\[ F = A(1+i)^3 + A(1+i)^2 + A(1+i) + A \]

\[ F = A(1+i)^{n-1} + A(1+i)^{n-2} + \ldots + A(1+i)^2 + A(1+i) + A \quad \ldots (1) \]

Multiply by \((1+i)\)

\[ (1+i)F = A(1+i)^n + A(1+i)^{n-1} + \ldots + A(1+i)^3 + A(1+i)^2 + A(1+i) \quad \ldots (2) \]

Subtract eqn (1) from eqn (2)

\[ (1+i)F - F = A(1+i)^n - A \]
\[(1+i)F - F = A(1+i)^n - A\]

\[i F = A[(1 + i)^n - 1]\]

\[F = A \left[ \frac{(1+i)^n - 1}{i} \right] = A\left(\frac{F}{A}, i\%, n\right)\]

Where \(A(F/A, i\%, n)\) is called **uniform series compound amount factor**
Also, we can solve for $A$ in terms of $F$

$$A = F \left[ \frac{i}{(1+i)^n - 1} \right] = F\left(\frac{A}{F}, i\%, n\right)$$

Where $F\left(\frac{A}{F}, i\%, n\right)$ is called *uniform series sinking fund factor*
Example 4-1

A man deposits $500 in a credit union at the end of each year for 5 years. The credit union pays 5% interest, compounded annually.

At the end of 5 years, immediately after the fifth deposit, how much does the man have in his account?

\[ A = $500, \quad i = 5\%, \quad n = 5 \]
Example 4-2

- Jim wants to save a uniform amount of money at the end of each month in order to save a $1000 at the end of each year.
- Bank pays 6% interest, compounded monthly.

How much would he have to deposit each month?

Given: $F = $1000, $i = \frac{6}{12} = \frac{1}{2}\%$, $n = 12$ months

$A =$ ?
Formula relating $A$ and $P$

$$A = F \left[ \frac{i}{(1+i)^n - 1} \right] = F\left(\frac{A}{F}, i\%, n\right)$$

$$A = P(1+i)^n \left[ \frac{i}{(1+i)^n - 1} \right]$$

$$A = P\left[ \frac{i(1+i)^n}{(1+i)^n - 1} \right] = P\left(\frac{A}{P}, i\%, n\right)$$

Where $P(A/P, i\%, n)$ is called uniform series capital recovery factor
Also, we can solve for $P$ in terms of $A$

$$P = A \left[ \frac{(1+i)^n - 1}{i(1+i)^n} \right] = A(P/A, i\%, n)$$

Where $A(P/A, i\%, n)$ is called uniform series present worth factor
Example 4-3

• You borrowed $5000 and want to repay in five equal end-of-year payments. The first payment is due one year after you receive the loan.
• Interest on the loan is 8% interest.

What is the size of each of the five payments?

i.e. with interest at 8%, a present sum of $5000 is equivalent to five equal end-of-period disbursements of $1252
Example 4-4

• An investor has a purchase contract on some machine tools.
• He will be paid $140 at the end of each month for a 5-year period.
• The investor offers to sell you the contract for a $6800 cash today.
• If you otherwise can make 1% per month on your money, would you accept or reject the investor’s offer.

\[ P = A\left(\frac{P}{A}, 1\%, 60\right) = 140(44.955) = 6293.7 \]

i.e. if you take the contract, you will be paid $140 per month for a period of 60 months. That means a total of $8400 over the 5-year period.

We need to determine if the contract is worth $6800.


... Example 4-4

- It is clear that if we pay $6800 for the contract and receive $140 per month, we will receive less than the 1% per month interest (which is the interest that we can otherwise make). Therefore the offer should be rejected.
- OR, we can say that the $140 per month payment is equivalent to $6293.7. Therefore if we pay $6800 for a benefit of $140 per month, we will lose money
- OR, we can say that the $6800 will give me more than $140 per month if invested at the given interest rate of 1%, which means that investing the $6800 in the purchase contract will be a loss when compared to investing it in in the other investment opportunity (which gives a 1% interest rate)

Therefore, Reject the offer.
Example 4-5

• Suppose that we decided to pay $6800 for the contract in example 4-4.
• What monthly rate of return would we obtain on our investment?

(See the text book for the solution)

Final answer: \( i = 0.722\% \), which means that the monthly rate of return on our investment is 0.722\% per month.
Example 4-6

- Using 15% interest rate, compute F
Example 4-7

- Using 15% interest rate, compute P
Relationships Between Compound Interest Factors

\[
P = \frac{F}{(1+i)^n} = F \left( \frac{P}{F}, i, n \right)
\]

\[
(F/P, i, n) = \frac{1}{(P/F, i, n)}
\]

\[
F = P(1+i)^n = P \left( \frac{F}{P}, i, n \right)
\]

\[
\frac{P}{A} = \left( 1 + \frac{i}{1+i} ight)^n - 1
\]

\[
(A/P, i, n) = \frac{1}{(P/A, i, n)}
\]

\[
F = A \left( \frac{(1+i)^n - 1}{i} \right) = A \left( \frac{F}{A}, i, n \right)
\]

\[
(A/F, i, n) = \frac{1}{(F/A, i, n)}
\]

\[
A = P \left( \frac{i}{(1+i)^n-1} \right) = F \left( \frac{A}{F}, i, n \right)
\]

\[
\frac{A}{P} = \frac{1}{(F/P, i, n)}
\]
Relationships Between Compound Interest Factors

\[(P / A, i, n) = \sum_{J=1}^{n} (P / F, i, J)\]

**How?**

\[P = A(1 + i)^{J-1} + A(1 + i)^{J-2} + \ldots + A(1 + i)^n\]

\[P = A[(1 + i)^{-1} + (1 + i)^{-2} + \ldots + (1 + i)^n]\]

\[P = A[(P/F, i, 1)+(P/F, i, 2)+\ldots+(P/F, i, n)]\]

since \(P = A(P/A, i, n)\)

We conclude that \((P/A, i, n) = (P/F, i, 1)+(P/F, i, 2)+\ldots+(P/F, i, n)\)

\[\Rightarrow (P / A, i, n) = \sum_{J=1}^{n} (P / F, i, J)\]

**For Example:**

\((P/A, 5\%, 4) = (P/F, 5\%, 1) + (P/F, 5\%, 2) + (P/F, 5\%, 3) + (P/F, 5\%, 4)\)
Relationships Between Compound Interest Factors

\[
(F / A, i, n) = 1 + \sum_{J=1}^{n-1} (F / P, i, J)
\]

\[
F = A + A(1+i) + A(1+i)^2 + \ldots + A(1+i)^{n-1}
\]

\[
F = A \left[ 1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{n-1} \right]
\]

\[
F = A \left[ 1 + (F/P, i, 1) + (F/P, i, 2) + \ldots + (F/P, i, n-1) \right]
\]

since \( F = A(F/A, i, n) \)

We conclude that \( (F/A, i, n) = 1 + (F/P, i, 1) + (F/P, i, 2) + \ldots + (F/P, i, n-1) \)

\[
\Rightarrow (F / A, i, n) = 1 + \sum_{J=1}^{n-1} (F / P, i, J)
\]
Relationships Between Compound Interest Factors

\[(A/P,i,n) = \frac{i(1 + i)^n}{(1+i)^n - 1}\]
\[(A/F,i,n) = \frac{i}{(1+i)^n - 1}\]

We start with an identity:

\[i (1+i)^n = i + i (1+i)^n - i = i + i [(1+i)^n - 1]\]

Now divide by \((1+i)^n - 1\) to get

\[\frac{i (1+i)^n}{(1+i)^n - 1} = \frac{i}{(1+i)^n - 1} + i.\]
This gives:

\[(A/P,i,n) = (A/F,i,n) + i\]
Arithmetic Gradient

Suppose you buy a car. You wish to set up enough money in a bank account to pay for standard maintenance on the car for the first five years. You estimate the maintenance cost increases by $G = $30 each year. The maintenance cost for year 1 is estimated as $120. Thus, estimated costs by year are $120, $150, $180, $210, $240.
Arithmetic Gradient

We break up the cash flows into two components:

\[ A = 120 \]

and

\[ G = 30 \]

\[ P_1 = A \left( \frac{P}{A}, 5\%, 5 \right) = 120 \left( \frac{P}{A}, 5\%, 5 \right) = 120 \times 4.329 = 519 \]

\[ P_2 = G \left( \frac{P}{G}, 5\%, 5 \right) = 30 \left( \frac{P}{G}, 5\%, 5 \right) = 30 \times 8.237 = 247 \]

\[ P = P_1 + P_2 = 766. \]

Note: 5 and not 4. Using 4 is a common mistake.
Arithmetic Gradient

Arithmetic Gradient Present Worth

\[(P/G,i,n) = \{ [(1+i)^n - i n - 1] / [i^2 (1+i)^n] \} \]

Arithmetic Gradient Uniform Series

\[(A/G,i,n) = \{ (1/i) - n / [(1+i)^n - 1] \} \]

\[(F/G,i,n) = G [(1+i)n-in-1]/i2 \]

\[(P/G,5\%,5) \]
\[= \{[(1+i)^n - i n - 1]/[i^2 (1+i)^n]\} \]
\[= \{[(1.05)^5 - 0.25 - 1]/[0.05^2 (1.05)^5]\} \]
\[= 8.23691676. \]
Arithmetic Gradient

**Example 4-6.** Maintenance costs of a machine start at $100 and go up by $100 each year for 4 years. What is the equivalent uniform annual maintenance cost for the machinery if $i = 6\%$.

This is not in the standard form for using the gradient equation, because the year-one cash flow is not zero.

We reformulate the problem as follows.
The second diagram is in the form of a $100 uniform series. The last diagram is now in the standard form for the gradient equation with \( n = 4, G = 100 \).

\[
A = A_1 + G \left( \frac{A}{G}, 6\%, 4 \right) = 100 + 100 \times 1.427 = 242.70
\]
Arithmetic Gradient

Example

With \( i = 10\% \), \( n = 4 \), find an equivalent uniform payment \( A' \) for

\[
\begin{align*}
24000 & \quad \text{1st year} \\
18000 & \quad \text{2nd year} \\
12000 & \quad \text{3rd year} \\
6000 & \quad \text{4th year}
\end{align*}
\]

This is a problem with decreasing costs instead of increasing costs.

The cash flow can be rewritten as the DIFFERENCE of the following two diagrams, the second of which is in the standard form we need, the first of which is a series of uniform payments.
Arithmetic Gradient

\[ A = A_1 - G(A/G, 10\%, 4) \]

\[ = 24000 - 6000 \times (A/G, 10\%, 4) \]

\[ = 24000 - 6000(1.381) = 15,714. \]
Arithmetic Gradient

Example  Find P for the following diagram with i = 10%.

This is not in the standard form for the arithmetic gradient. However, if we insert a “present value” J at the end of year 2, the diagram *from that point on* is in standard form.

Thus:

- \( J = 50 \times (P/G, 10\%, 4) = 50 \times 4.378 = 218.90 \)
- \( P = J \times (P/F, 10\%, 2) = 218.90 \times 0.8264 = 180.9 \)

*OR in one line:* \( P = 50 \times (P/G, 10\%, 4) \times (P/F, 10\%, 2) = 50 \times 4.378 \times 0.8264 \)
Geometric Gradient

**In arithmetic gradient**, the period-by-period change is a uniform **amount**, \( G \).

**In geometric gradient**, the period-by-period change is a uniform **rate**, \( g \).

Hence we can define the **geometric gradient** \((g)\) as a uniform rate of cash flow increase/decrease from period to period.

Example: Suppose you have a vehicle. The first year maintenance cost is estimated to be **$100**. The rate of increase in each subsequent year is **10%** \((g)\). You want to know the present worth of the cost of the first five years of maintenance, given \( i = 8\% \).
Derivation of Geometric Gradient Formula

The cost in any year is

\[ A_n = A_1(1+g)^{n-1} \]

*Fort example: the cost in the third year is \( A_3 = A_1(1+g)^2 \)*

Where: \( g \) = geometric gradient uniform rate of cash flow increase/decrease  
\( A_1 \) = Value of cash flow at year 1  
\( A_n \) = Value of cash flow at any year \( n \)

The present worth of any cash flow is

\[ P_n = A_n(1+i)^{-n} \]

\[ P_n = A_1(1+g)^{n-1}(1+i)^{-n} \]

\[ P_n = A_1(1+g)^{n-1}(1+i)^{-n} \cdot [(1+i)(1+i)^{-1}] \]

\[ P_n = A_1(1+g)^{n-1}(1+i)^{-n} \cdot (1+i)^{-1} \]
Derivation of Geometric Gradient Formula

\[ P_n = A_i (1+i)^{-1} \frac{(1+g)^{n-1}}{(1+i)^{n-1}} \]

\[ P_n = A_i (1+i)^{-1} \left[ \frac{1}{1+i} \right]^{n-1} \]

\[ P = A_i (1+i)^{-1} \sum_{x=1}^{n} \left[ \frac{1}{1+i} \right]^{x-1} \]

\[ P = A_i (1+i)^{-1} + A_i (1+i)^{-1} \left( \frac{1+g}{1+i} \right) + A_i (1+i)^{-1} \left( \frac{1+g}{1+i} \right)^2 + \ldots + A_i (1+i)^{-1} \left( \frac{1+g}{1+i} \right)^{n-1} \]

Let \( a = A_i (1+i)^{-1} \) and \( b = \frac{(1+g)}{(1+i)} \)

\[ P = a + ab + ab^2 + \ldots + ab^{n-1} \quad - - - - - - (1), \] Multiply the eqn by \( b \) to get

\[ bP = ab + ab^2 + ab^3 + \ldots + ab^n \quad - - - - - - - - (2) \]

Subtracting eqn (2) from eqn (1) yields

\[ P - bP = a - ab^n \Rightarrow P = \frac{a(1-b^n)}{1-b} \]

Replacing the original values for \( a \) and \( b \), we obtain ......
... Derivation of Geometric Gradient Formula

\[ P = A_1 (1 + i)^{-1} \frac{1 - \left(1 + \frac{g}{1+i}\right)^n}{1 - \left(1 + \frac{g}{1+i}\right)} = A_1 \frac{1 - \left(1 + \frac{g}{1+i}\right)^n}{(1+i) - \left(1 + \frac{g}{1+i}\right)(1+i)} \]

\[ = A_1 \left[ \frac{1 - (1 + g)^n (1+i)^{-n}}{1 + i - 1 - g} \right] \]

\[ P = A_1 \left[ \frac{1 - (1 + g)^n (1+i)^{-n}}{i - g} \right], \text{ where } i \neq g \]

The expression in brackets is the geometric series present worth factor.

If \( i = g \), use \( P = A_1 n(1 + i)^{-1} \)
Example 4-12

The first-year maintenance cost for a new automobile is estimated to be $100, and it increases at a uniform rate of 10% per year. Using an 8% interest rate, calculate the present worth of cost of the first 5 years of maintenance.

Solution

\[
P = A_l \left[ \frac{1-(1+g)^n(1+i)^{-n}}{i-g} \right]
\]

\[A_l = \$100, \ g = 10\%, \ i = 8\%\]

\[
P = 100 \left[ \frac{1-(1.10)^5(1.08)^{-5}}{0.08-0.10} \right] = \$480.42
\]
Nominal & Effective Interest

A) Consider the situation of a person depositing $1000 into a bank that pays 12% interest, compounded monthly.

How much would be in the savings account at the end of one year for case (A) and case (B)

12% interest, compounded monthly, means that the bank pays 1% every month.

\[
i = 1\%, \quad n = 12, \quad P = $1000
\]

\[
F = 1000(F/P, 1\%, 12)
\]

\[
= 1000(1.01)^{12} = $1126.8
\]

B) Consider the situation of a person depositing $1000 into a bank that pays 12% interest, compounded annually.

12% interest, means that the bank pays 12% every year.

\[
i = 12\%, \quad n = 1, \quad P = $1000
\]

\[
F = 1000(F/P, 12\%, 1)
\]

\[
= 1000(1.12^1) = $1120.0
\]
Nominal & Effective Interest

- **Nominal interest rate** per year, \( r \), is the annual interest rate without considering the effect of any compounding. 
  
  *(it is the 12% in the previous example)*

- **Effective interest rate** per year, \( i_a \), is the annual interest rate taking into account the effect of any compounding during the year.
  
  *(In the previous example, \( i_a = 126.8/1000 = 12.68\%)*

- **Effective interest rate** per interest period, \( i \).
  
  *(it is the 1% used in the previous example)*

- \( m = \) Number of compounding subperiods per time period
  
  *(It was the 12 compounding periods used in the previous example)*
Derivation of the effective interest rate equation

If a $1 deposit was made to an account that compounded interest $m$ times per year and paid a nominal interest, $r$:

** ► ► Interest rate per subperiod = $r/m$

Total in the account at the end of year = $1(1+r/m)^m$

We can determine the effective interest rate by deducting the $1 principal amount

Therefore, effective interest rate per year is

$$i_a = \left(1 + \frac{r}{m}\right)^m - 1$$

OR just substitute $i = r/m$ to get

$$i_a = (1 + i)^m - 1$$
Nominal & Effective Interest

Common nomenclature in engineering Economics
When we say “the interest rate is 6% compounded monthly”, we mean:

1- \( r = 6\% \) per year \quad \text{(nominal interest rate)}
2- \( i = \frac{r}{m} = 6/12 = 0.5\% \) per month \quad \text{(interest per period)}
3- \( i_a = (1+r/m)^m - 1 \quad \text{(Effective interest rate per year)} \)
   \[ = (1+0.005)^{12} - 1 = 6.1678\% \]

When we say interest rate is 6\% per month, we mean:

1- \( i = 6\% \) per month \quad \text{(interest per period)}
2- \( r = 72\% \) per year \quad \text{(nominal interest rate)}
3- \( i_a = (1+r/m)^m - 1 \quad \text{(Effective interest rate per year)} \)
   \[ = (1+0.06)^{12} - 1 = 101.22\% \]
Example 4-14

If a savings bank pays 1½% interest every 3 months, what are the nominal and effective interest rates per year?

\( i = 1 \frac{1}{2}\% \)  (effective interest rate per interest period)
\( m = 4 \)  (number of compounding subperiods per time period)

Nominal interest rate per year \( r = 4 \times 1 \frac{1}{2}\% = 6\% \)

Effective interest rate per year \( i_a = \left(1 + \frac{r}{m}\right)^m - 1 \)
\[ i_a = \left(1 + \frac{0.06}{4}\right)^4 - 1 = 0.061 = 6.1\% \]

OR Effective interest rate per year \( i_a = (1 + i)^m - 1 \)
\[ = (1 + 0.015)^4 - 1 = 0.061 \]
\[ = 6.1\% \]
Example 4-15

A loan shark lends money on the following terms: “If I give you $50 on Monday, you owe me $60 on the following Monday:

(a) What nominal interest rate per year \( r \) is the loan shark charging?
(b) What effective interest rate per year \( i_a \) is he charging?
(c) If the loan shark started with $50, how much money he would have at the end of the year?

a) **Nominal interest rate per year \( r \)?**

First we need to calculate the interest rate per period (week)

\[
F = P(F/P, i, n)
\]

\[
60 = 50(F/P, i, 1), \text{ therefore, } (F/P, i, 1) = 1.2 \quad \Rightarrow \quad i = 20\% \text{ per week}
\]

*Nominal interest rate per year* = 52 weeks \( \times 0.20 = 10.4 = 1040\% \)
(b) Effective interest rate per year \((i_a)\)?

\[
i_a = \left(1 + \frac{r}{m}\right)^m - 1 = \left(1 + \frac{10.40}{52}\right)^{52} - 1 = 13,105 - 1 = 13,104 = 1,310,400\%
\]

OR

\[
i_a = (1 + i)^m - 1 = (1 + 0.20)^{52} - 1 = 13,104 = 1,310,400\%
\]

(c) How much at the end of the year?

\[
F = P(1 + i)^n = 50(1 + 0.20)^{52} = $655,200
\]
Example 4-16
On January 1, a woman deposits $5000 in a credit union that pays 8% nominal annual interest, compounded quarterly. She wishes to withdraw all the money in five equal yearly sums, beginning December 31 of the first year. How much should she withdraw each year?

Nominal interest rate $r = 8\%$ compounded quarterly.
Therefore, the effective interest rate per interest period $i = 2\%$

In example 4-3, we used $A = P(A/P, i, n)$. Can we do the same thing here? We can’t apply it directly since the compounding period does not match the annual withdrawals.
**Example 4-16**

**Solution 1**

Compute the effective interest $i_a$ per year

$$i_a = (1 + \frac{r}{m})^m - 1 = (1 + 0.08/4)^4 - 1 = 0.0824 = 8.24\%$$

$W = 5000(A/P, 8.24\%, 5)$

Use A/P formula since 8.24\% does not exist in tables
Example 4-16
Solution 2
Compute the equivalent uniform cash flows, A, at the end of each quarter

\[ A = 5000 \times (A/P, 2\%, 20) = 5000 \times (0.0612) = 306 \]

Now, we can calculate W from A for each one year period

\[ W = A \times (F/A, 2\%, 4) = 306 \times (4.122) = 1260 \]
Continuous Compounding

In all previous examples, we used periodic compounding, where the duration of the interest period was a finite number (e.g. a year, six months, one month, one week, … etc.

<table>
<thead>
<tr>
<th>Nominal rate</th>
<th>Yearly</th>
<th>Semi-ann.</th>
<th>Monthly</th>
<th>Daily</th>
<th>Continuously</th>
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<td>m = 12</td>
<td>m = 365</td>
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</tr>
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</table>

By continuous compounding, we mean to increase the number of compounding periods (m) to infinity.
Continuous Compounding

\[ F = P(1 + i)^n \]

If we have \( m \) compounding subperiods in the year, then

\[ F = P \left(1 + \frac{r}{m}\right)^{mn} \quad (mn \text{ will be the number of subperiods in } n \text{ years}) \]

To obtain a formula corresponding to continuous compounding, we need to increase \( m \) and make it very large; i.e. \( m \to \infty \)

For continuous compounding:

\[ F = P \cdot \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{mn} \]

An important limit in calculus is

\[ \lim_{x \to 0} (1 + x)^{1/x} = 2.71828 = e \]

Therefore, set \( x = r/m \), and \( mn \) becomes \((1/x)(rn)\)

\[ F = P \cdot \left[\lim_{x \to 0} (1 + x)^{1/x}\right]^m \]

\[ F = P \cdot (1 + i)^n \quad \text{becomes} \quad F = Pe^{rn} \]

\[ P = Fe^{-rn} \]

Therefore, for continuous compounding,

\[ (1 + i) = e^r \]

Effective interest rate per year for continuous compounding

\[ i_a = e^r - 1 \]
Continuous Compounding

The following formulas apply for continuous compounding. They are obtained by substituting $i = e^r - 1$ into the previously studied formulas.

- **Compound Amount**
  \[ F = Pe^{rn} = P[F / P, r, n] \]

- **Present Worth**
  \[ P = Fe^{-rn} = F[P / F, r, n] \]

- **Sinking Fund**
  \[ A = F\left[ \frac{e^r - 1}{e^{rn} - 1} \right] = F[A / F, r, n] \]

- **Capital Recovery**
  \[ A = P\left[ \frac{e^{rn}(e^r - 1)}{e^{rn} - 1} \right] = P[A / P, r, n] \]

- **Series Compound Amount**
  \[ F = A\left[ \frac{e^{rn} - 1}{e^r - 1} \right] = A[F / A, r, n] \]

- **Uniform Series Present Worth**
  \[ P = A\left[ \frac{e^{rn} - 1}{e^{rn}(e^r - 1)} \right] = A[P / A, r, n] \]
Example 4-17

If you were to deposit $2000 in a bank that pays 5% nominal interest, compounded continuously, how much would be in the account at the end of 2 years?

\[ F = Pe^{rn} \]

\[ P = $2000, \quad r = \text{nominal interest rate} = 0.05 \]
\[ n = \text{number of years} \]

\[ F = 2000e^{(0.05)(2)} = 2000(1.1052) = $2210.40 \]
Example 4-18

A bank offers to sell savings certificates that will pay the purchaser $5000 at the end of 10 years but will pay nothing to the purchaser in the meantime. If interest is computed at 6%, compounded continuously, at what price is the bank selling the certificates?

\[ P = F e^{-rn} \]

\[ F = $5000, \quad r = \text{nominal interest rate} = 0.06 \]

\[ n = \text{number of years} = 10 \]

\[ P = 5000e^{-(0.06)(10)} = 5000(0.5488) = $2744 \]

Therefore, the bank is selling the $5000 certificates for $2744.
Example 4-20

If a savings bank pays 6% interest, compounded continuously, what are the nominal and the effective interest rates?

Nominal interest rate = 6% per year

Effective interest rate = $e^r - 1 = e^{0.06} - 1 = 0.0618 = 6.18\%$