The Islamic University of Gaza
Faculty of Engineering
Civil Engineering Engineering Department
Infrastructure Msc.

Special Topics in Water and Environment - ENGC 6383

Lecture 5

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A groundwater system is essential three dimensional, consisting of:

- water bearing layers (aquifers),
- The flux exchange between layers is determined by the piezometric heads and the horizontal and vertical permeability.
- the separating layers are continuously and impervious then each water bearing layer or aquifer is a groundwater system in itself.
Equation and Analytical Solutions

Groundwater Flow

**Groundwater System**

Although groundwater flow and transport is essential three-dimensional, groundwater systems may be schematized to two and one-dimensional systems.

- A three dimensional flow approach (3D) is applied in advanced and detailed studies, in particular for the determination of flow patterns in relation to convective flow and related advective solute transport.

- Two-dimensional areal flow (2D) is usually characterized in the horizontal x-y plain. The flow in the vertical z-direction is neglected.
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GENERAL FLOW EQUATIONS

THREE DIMENSIONAL FLOW EQUATION

The groundwater flow or flux mass balance consists of three major components

1. **Induced flow** by the spatial differences in hydraulic head, usually called Darcy flow.

2. **Forced flow** is represented as volumetric external inflow from sources and sinks.

3. **Storage flow**, induced by a change of water levels or piezometric head

![Fig 3.4 General mass balance](image-url)
GENERAL FLOW EQUATIONS

THREE DIMENSIONAL FLOW EQUATION

1. Induced flow or Darcy Flow

\[ q_i = - k_i \frac{\delta \phi}{\delta x_i} \]

\[ q_i = - k_i \frac{\Delta \phi}{\Delta x_i} \]

where
- \( q_i \) = specific discharge in i-direction (m³/m²/day)
- \( k_i \) = hydraulic conductivity in i-direction (m/day)
- \( \phi \) = hydraulic head (m)
- \( \Delta \phi \) = difference in head over \( \Delta x_i \), (m)

The net Darcy inflow in x-direction

\[ q_x = (q_x + \frac{\delta q_x}{\delta x} \Delta x) \Delta y \Delta z = - \frac{\delta q_i}{\delta x_i} \Delta v \]

3D the darcy inflow

\[ \frac{\delta}{\delta x_j} \left( k_i \frac{\delta \phi}{\delta x_j} \right) \Delta v \text{ or } \nabla \left( k \nabla \phi \right) \cdot \Delta V \]

with \( v = \frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z} \)
GENERAL FLOW EQUATIONS

THREE DIMENSIONAL FLOW EQUATION

Source and Sink

Sources: are external forced inflows like natural percolation from the unsaturated zone and artificial recharge from surface waters and recharge wells.

Sinks are outflows from the control volume, like natural drainage and well discharges

\[(q_{31} - q_{30}) \cdot \Delta V\]

Where \(\Delta V\) is the control volume.

Fig 3.4 General mass balance
GENERAL FLOW EQUATIONS

THREE DIMENSIONAL FLOW EQUATION

Storage Flow

The change in piezometric level induces a change in storage.

\[ S \cdot \frac{\partial \phi}{\partial t} \cdot \Delta V \]

Where, \( S \) is the specific storage

The 3D general flow mass balance:

\[ -\left( \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) + (q_{3i} - q_{3o}) - S \frac{\partial \phi}{\partial t} = 0 \]

Substituting Darcy for \( x_i = x,y,z \)

\[ q_i = -k_i \frac{\partial \phi}{\partial x_i} \]
GENERAL FLOW EQUATIONS

THREE DIMENSIONAL FLOW EQUATION

The 3D general flow mass balance:

\[-\left(\frac{\delta q_x}{\delta x} + \frac{\delta q_y}{\delta y} + \frac{\delta q_z}{\delta z}\right) + \left(q_{3_i} - q_{3_o}\right) - S\frac{\delta \phi}{\delta t} = 0\]

Substituting Darcy for \( x_i = x, y, z \)

\[q_i = - k_i \frac{\delta \phi}{\delta x_i}\]

\[\frac{\delta}{\delta x}\left(k_x \frac{\delta \phi}{\delta x}\right) + \frac{\delta}{\delta y}\left(k_y \frac{\delta \phi}{\delta y}\right) + \frac{\delta}{\delta z}\left(k_z \frac{\delta \phi}{\delta z}\right) + \left(q_{3_i} - q_{3_o}\right) - S\frac{\delta \phi}{\delta t} = 0\]

\[\nabla \cdot k \cdot (\nabla \phi) + \left(q_{3_i} - q_{3_o}\right) - S\frac{\delta \phi}{\delta t} = 0\]

Fig 3.4 General mass balance
GENERAL FLOW EQUATIONS

The 2D case the aquifer thickness $H$ is accounted for:

$$\frac{\delta}{\delta x} \left( k_x H \frac{\delta \Phi}{\delta x} \right) + \frac{\delta}{\delta y} \left( k_y H \frac{\delta \Phi}{\delta y} \right) + (q_{2i} - q_{2o}) - S_s \frac{\delta \Phi}{\delta t} = 0$$

where $q_{2i}$ and $q_{2o}$ (m3/m2/d) are aerial fluxes like percolation and seepage through confining layers,

and $S_s$ is the elastic storage (storage coefficient) for confined aquifers.

Fig 3.6 Two-dimensional areal block
**Recharge** is the forced external inflow into the groundwater system.

Recharge can be specified as
- a fixed rate at a specific location (m$^3$/day),
- a linear inflow along a line (m$^3$/m/d) or as
- a diffuse inflow over a specific area (m$^3$/m$^2$/d).

**Storage flow** in a phreatic aquifer (unconfined) results from the filling of the void space in fractures and fissures (rock) or in between the grains (soil)

\[
\Delta S = A \ S_y \Delta h
\]

In confined aquifer, the change in storage is determined as:

\[
\Delta S = A \ S_s \Delta \phi
\]
The general 2-D groundwater flow is

\[-\left(\frac{\delta q_x}{\delta x} + \frac{\delta q_y}{\delta y}\right) + q_2^{ext} - S\frac{\delta \phi}{\delta t} = 0\]

\[\left[\frac{\delta}{\delta x}\left(k_x H \frac{\delta \phi}{\delta x}\right) + \frac{\delta}{\delta y}\left(k_y H \frac{\delta \phi}{\delta y}\right)\right] + (q_{2i} - q_{2o}) - S\frac{\delta \phi}{\delta t} = 0\]

where

\[q_{2i} = \text{areal external inflow (m3/m2/d)}\]
\[q_{2o} = \text{areal external outflow (m3/m2/d)}\]

\[\nabla \cdot (kH \nabla \phi) + q_2 - S\frac{\delta \phi}{\delta t} = 0\]
Leakage or seepage is induced flow through semi-permeable layers, in general the interaction between aquifers.

Or Interaction between surface water bodies (rivers, lakes) and the groundwater system is also an induced flow, the reverse process is called drainage

\[ q' = - k' \frac{\phi - \phi'}{D'} = - \frac{\phi - \phi'}{C} \]

where
- \( q' \) = discharge or flux to an aquifer through a separating layer (m³/day)
- \( \phi, \phi' \) = hydraulic head in resp. considered and bounding aquifer (m)
- \( k' \) = vertical conductivity of separating layer (m)
- \( D' \) = thickness of separating layer (m)
- \( C \) = hydraulic resistance of the separating layer (days)

For Unconfined aquifer

\[
\left[ \frac{\delta}{\delta x} (k_x h \frac{\delta h}{\delta x}) + \frac{\delta}{\delta y} (k_y h \frac{\delta h}{\delta y}) \right] - \frac{h - \phi'}{C} + (q_{2z} - q_{2o}) - s_y \frac{\delta h}{\delta t} = 0
\]

Fig 3.8 Leakage in semi-confined aquifers
Interaction with surface water is essential induced flow

\[ Q_{\text{surf}} = -A \frac{\phi - \phi_{\text{surf}}}{C} \quad \text{if } \phi > \phi_{\text{bot}} \]

\[ Q_{\text{surf}} = -A \frac{\phi_{\text{bot}} - \phi_{\text{surf}}}{C} \quad \text{if } \phi < \phi_{\text{bot}} \]

where

- \( Q_{\text{surf}} \) = induced recharge from the surface water (m³/day)
- \( A \) = area of infiltrating water body (m²)
- \( \phi_{\text{surf}} \) = hydraulic head of surface water (m)
- \( \phi_{\text{bot}} \) = level of the surface water (m)
- \( C \) = hydraulic resistance between surface and aquifer (days)

Fig 3.9 Induced surface water recharge
SOLUTION METHODS

Modelling is the representation of the physical reality by means of analogons, physical scale models and mathematical methods.

Selection of solution methods in mathematical modelling of groundwater flow and transport depends on the physical conditions, complexity of the problem and preference of the user.

1. **Analytical methods or 'exact' solutions**, where the hydraulic head, flow (q, v) and concentrations of constituents (c) are solved by integration in time and space. (homogeneous and uniform problems.)

2. **Semi-analytical methods**, where general analytical solutions are superimposed and finally solved in time by numerical integration. simplified two-dimensional areal cases applied in preliminary design studies.
SOLUTION METHODS

3. **Numerical methods** where the hydraulic head and flow are computed in time and space at discrete points of a projected network. Provide mathematical solutions for *almost all groundwater flow problems*, and is most applicable in non-homogeneous and multi-dimensional flow.

4. **Boundary element methods and analytical element methods** where conditions specified at the boundaries fit the real boundary conditions. Their complexity limit their application to real world problems.
Modelling Errors

When making modelling, errors are introduced. These errors relate to:
- the simplification of reality,
- basic assumptions,
- data accuracy,
- schematization and selected computational method.

Types of errors can be subdivided as:

- **physical system errors** introduced by the simplification and schematization of the physical reality to a conceptual model and related parameters
- **mathematical errors** by expressing the physical system behaviour by differential equations
- **numerical errors** due to transform of differential into numerical equations
- **computational errors** due to convergence and computer inaccuracies.
Analytical Methods

**Analytical methods** provide
-an quick solution for the simplified problem,
- often serves as a verification of solutions of more complex problems obtained by numerical computer oriented methods.

-For the applications of these methods see examples 3.4.1, 3.4.2, **3.4.3** and 3.4.4
Example 3.4.3 Steady flow in a uniform confined aquifer, including infiltration.

\[ \begin{align*}
\phi_o & = +20 \text{ m} \\
\phi_L & = +14 \text{ m} \\
k & = 10 \text{ m/d} \\
H_o & = 20 \text{ m} \\
L & = 1000 \text{ m} \\
I & = 1 \text{ mm/d}
\end{align*} \]

a. Find a relation between the discharge \( q \) and the distance \( x \).
b. Find a relation between the piezometric head \( \phi \) and the distance \( x \).
c. Compute the piezometric level and flow at \( x=0, L/2 \) and \( L \).
d. Determine the flow to the lake at \( x=0 \) and \( x=L \).
Example 3.4.3 Steady flow in a uniform confined aquifer, including infiltration.

Solution

Continuity: \[ q + I \cdot \delta x = q + \delta q \quad \Rightarrow \quad \delta q = I \cdot \delta x \]

Integration: \[ q = Ix + C_1 \]

Darcy: \[ q = -kH_o \frac{\delta \phi}{\delta x} \]

and \[ Ix + kH_o \frac{\delta \phi}{\delta x} + C_1 = 0 \]

Integration yields the general equation:

\[ \frac{Ix^2}{2} + kH_o \phi + C_1 x + C_2 = 0 \]

Boundary conditions:

\[ x = 0, \phi = \phi_o \quad \Rightarrow \quad C_2 = -kH_o \phi_o \]

\[ x = L, \phi = \phi_L \quad \Rightarrow \quad C_1 = -\frac{IL}{2} - kH_o \frac{\phi_L - \phi_o}{L} \]

which results in:

a) \[ q = I(x - \frac{L}{2}) - kH_o \frac{\phi_L - \phi_o}{L} \]

b) \[ \phi = \phi_o + I \frac{Lx - x^2}{2kH_o} + \frac{\phi_L - \phi_o}{L} x \]
Example 3.4.3 Steady flow in a uniform confined aquifer, including infiltration.

c) Substituting data yields:

\[ q = I\left(x - \frac{L}{2}\right) - kH_o \frac{\phi_L - \phi_o}{L} = -\frac{1}{2} - 10.20 \cdot \frac{14 - 20}{1000} = -0.5 + 1.2 = 0.7 \text{m}^3/\text{m/d} \]

\[ \phi = \phi_o + I \frac{L^2}{8kH_o} - \frac{\phi_L - \phi_o}{2} = \]

\[ 20 + \frac{1}{1000} \frac{1000^2}{8(10)20} - \frac{14 - 20}{2} = 20 + 0.63 - 3 = 17.63 \text{m} \]

\[ x = L \]

\[ \phi = \phi_L = +14 \text{ m} \]

\[ q = I\left(\frac{L}{2}\right) - kH_o \frac{\phi_L - \phi_o}{L} = \]

\[ = 10 \cdot \frac{1000}{2} - 10.20 \cdot \frac{14 - 20}{1000} = 0.5 + 1.2 = 1.7 \text{ m}^3/\text{m/d} \]

e) Lake interaction

\[ x = 0 \]

\[ q_{lake} = -q_o = -0.7 \text{ m}^3/\text{m/d} \]

\[ x = L \]

\[ q_{lake} = q_L = 1.7 \text{ m}^3/\text{m/d} \]
NUMERICAL METHODS

Applied In:
- **complex problems**, especially under varying conditions in space (inhomogeneous hydrogeological conditions) and in time (varying boundary conditions, unsteady flow), numerical methods have proven their general applicability.

- Convective transport and related advective solute contaminant transport can be simulated using numerical methods,

- but particle tracking like the method of characteristics are preferred for reasons of numerical stability.
NUMERICAL METHODS

In groundwater modeling three methods are applied:

- Finite Difference Methods (FDM), including cell or block centred methods
- Finite Element Methods (FEM)
- Polygon based Difference or Integrated methods.

In these methods:

- the dependent variables, (the head and concentrations), are defined at discrete points (grid system) in space and time, where analytical solutions provide continuous solutions.
NUMERICAL METHODS

-the solution is an approximation to the exact solution, and its accuracy depends on the selected numerical procedure and the specification of the hydrogeological parameters.

*The smaller the user selected steps in space and time, the more detailed and exact the obtained solution.*

-In numerical modeling the domain is overlain by a rectangular, triangular or quadrangular grid

-Around each node an impact area or cell can be defined for which the node is representative.
NUMERICAL METHODS

Fig 4.1  Finite difference grid

Fig 4.2  Finite elements

Cell or block centred grid

Polygon based
Solution Procedures

-For each node and related impact area the governing differential equation, based on conservation of mass, is transformed to a difference equation which relates the hydraulic head at the node itself and the heads at the surrounding nodes. (Discretization).

-Depending on the selected scheme the set of equations can be independent of each other (explicit) or interrelated (implicit).

-Explicit methods are solved using standard substitution methods. For reasons of stability and accuracy, they are not often applied.
Implicit Solution Procedures

Solution methods can be divided into:

1. **Direct solutions**, where the set of equations is solved as a complete matrix (Gaussian elimination, Cholesky decomposition and Matrix inversion). >>> require large storage capacity and long computational time.

2. Indirect or iterative solutions, *where equation after equation is solved, using trial start values*. The speed and accuracy of the solution procedure highly depends on these initial values. Computed values are corrected until sufficient accuracy is reached. (Gauss-Seidl substitution, Successive over-relaxation and Conjugate Gradient methods).
Implicit Solution Procedures

Numerical models generate errors concerning
1. **Accuracy** defining how well the final difference equation represents the actual physical situation.

2. **Efficiency** concerning the computational effort in setting up the model and computation time.

3. **Stability** addressing to have a solution at all. Implicit numerical schemes like cell methods and FEM in general show a high rate of stability.

*Stability is highly dependent on the user selected time and space interval*
General Guide Lines for Grid System

A critical step in modelling is the design of a grid. The finer the grid, the more accurate the solution.

- Define grid nodes at well locations or the centre of a well field. Nodal parameters must be representative for the related impact area.

- Define the boundary accurately, even if the grid has to be expanded to cover distant boundaries.

- Avoid large spacings next to small ones.

- Refine the grid in areas with a rapid spatial changes in heads and geohydrologic parameters like transmissivity.